CHAPTER 2

BASIC STRUCTURES:
SETS AND FUNCTIONS

Outline

- **Content**
  - Sets
  - Set Operations
  - Functions
  - Sequences and Summations
- **Reading**
  - Chapter 2

Most of the following slides are by courtesy of Prof. J.-D. Huang, Prof. H.-C. Yen, Prof. M.P. Frank

3 Sets

Sets

- **A set is an unordered collection of objects.**
- **The objects in a set are also called the elements or members of the set. A set is said to contain its elements.**

∈lement

- **Notation**
  - Membership in sets: $a \in A$; $a \notin A$
  - Set:
    - List all the members of the set
      - The set *V* of all vowels: $V = \{a, e, i, o, u\}$
    - List part of the members and ellipses
      - Positive integers less than 100: {1, 2, 3, …, 99}
    - Use set builder notation
      - $\{x|P(x)\}$ is the set of all *x* such that $P(x)$.
      - $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$
        $= \{1, 3, 5, 7, 9\}$

Basic Properties of Sets

- **Sets are inherently unordered:**
  - No matter what objects a, b, and c denote,
    $\{a, b, c\} = \{a, c, b\} = \{b, a, c\} = \{b, c, a\} = \{c, a, b\} = \{c, b, a\}$.
- **All elements are distinct (unequal); multiple listings make no difference!**
  - If $a=b$, then $\{a, b, c\} = \{a, c\} = \{b, c\} = \{a, a, b, a, b, c, c, c, c\}$.
  - This set contains at most 2 elements!

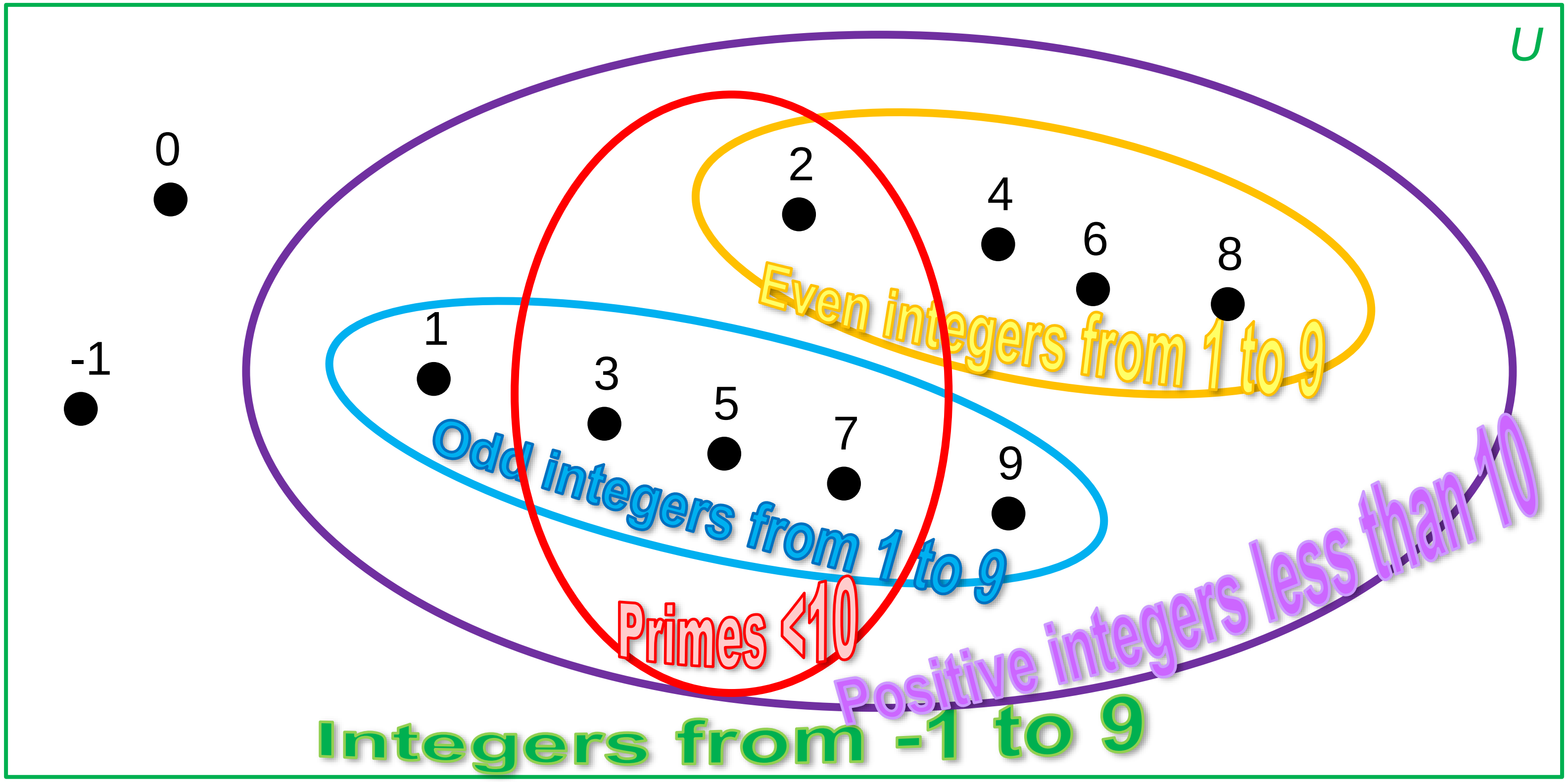
Useful Sets

- **We will use the following symbols to represent their respective sets:**
  - **N** = {0, 1, 2, …}, natural numbers
    - 0 is not considered as a natural number in some books.
  - **Z** = {…, -2, -1, 0, 1, 2, …}, integers
  - **Z**$^{+}$ = {1, 2, 3, …}, positive integers
  - **Q** = {*p*/*q* | *p* $\in$ **Z**, *q* $\in$ **Z**, and *q* $\neq$ 0}, rational numbers
  - **R**, real numbers, e.g., 374.1828471929498181 9…

Boldface letter

Venn Diagrams

- **The universal set contains all the objects under consideration**
  - Rectangle: the universal set
  - Circle: a set
  - Point: an element



Equality

- **Two sets are equal iff they have the same elements.**
- **E.g.,**
  - {1, 3, 5} = {3, 1, 5} = {3, 5, 1} = {1, 1, 3, 3, 5, 5, 5}
- **It does not matter how the set is defined or denoted.**
- **E.g.,**
  - {1, 2, 3, 4}

  = {x | x is an integer where $x > 0$ and $x < 5$ }

  = {x | x is a positive integer whose square is > 0 and < 25}

Empty Sets

- **An empty set (null set) is a set that has no elements, write ∅, or { }**

∅

- **A common error: ∅ vs. {∅}**
  - {∅} is **NOT** an empty set, {∅} ≠ ∅
  - {∅} is a singleton set (it has one element ∅)
- **Analogy: set :: folder**
  - ∅ :: an empty folder
  - {∅} :: a folder with exactly one folder inside, the empty folder

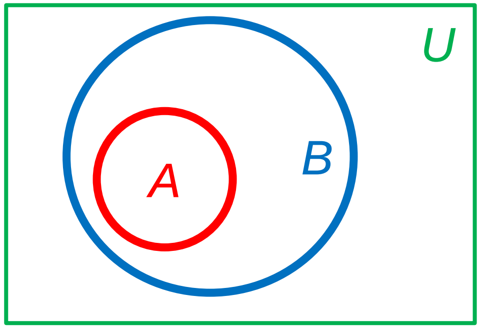
Subsets

- **The set A is a subset of set B, write $A \subseteq B$, iff every element of A is also an element of B**
  - i.e., $A \subseteq B$ iff $\forall x(x \in A \rightarrow x \in B)$ is true

- **Theorem: For any (nonempty) set S**
  1. **$\emptyset \subseteq S$, and**
  2. **$S \subseteq S$**

- **The set A is a proper subset of set B, write $A \subset B$, if $A \subseteq B$ but $A \neq B$**
  - i.e., $\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$

- **The set B is said to be a superset of set A iff every element of A is also an element of B**
  - i.e., $B \supseteq A \equiv A \subseteq B$



Proof of Set Equality

- **One useful way to show two sets are equal**
  If $A \subseteq B$ and $B \subseteq A$, then $A = B$
  - $\forall x(x \in A \rightarrow x \in B)$ and $\forall x(x \in B \rightarrow x \in A) \equiv \forall x(x \in A \leftrightarrow x \in B)$

Finite vs. Infinite

- **Let *S* be a set. If there are *n* distinct elements in *S* ($0 \le n < \infty$), we say *S* is a finite set and *n* is the cardinality of *S*. Write $|S|=n$.**
- **E.g.,**
  - Q: $|\varnothing| = ?$
  - A:
  - Q: $|\{\{1,2,3\},\{4,5\}\}| = ?$
  - A:

- **A set is infinite if it is not finite.**
- **E.g.,**
  - Symbols for some special infinite sets: **N, Z**, **R**
- **Infinite sets come in different sizes!**

Power Sets

- **Given a set S, the power set P(S) of S is the set of all subsets of S. i.e., P(S) = $\{x \mid x \subseteq S\}$.**
- **E.g.**
  - P($\{a, b\}$) = $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
  - P($\emptyset$) = $\{x \mid x \subseteq \emptyset\} = \{\emptyset\}$
  - P($\{\emptyset\}$) = $\{x \mid x \subseteq \{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$
- **Sometimes P(S) is written 2^S.**
  Note that for a finite set S, |P(S)| = $2^{|S|}$.
- **It turns out that |P(N)| > |N|.**
  There are different sizes of infinite sets!

Ordered N-Tuples

- **An ordered *n*-tuple (a_1, a_2, …, a_n) is the ordered collection that has a_1 as its first element, …, and a_n as its n^{th} element.**
  - C.f. {…}: unordered; (…): ordered
  - {1, 2} = {2, 1} = {2, 1, 1} vs. (1, 2) ≠ (2, 1) ≠ (2, 1, 1)
- **Two *n*-tuples *A* and *B* are equal iff $a_i = b_i$, for $i = 1, 2, \ldots, n$.**
- **Naming:**
  - Originated from single, double, triple, quadruple, quintuple, sextuple, septuple, octuple, ..., *n*-tuples
  - 2-tuples: pairs
  - 3-tuple: triple
  - 4-tuple: quadruple
  - 5-tuple: quintuple

Cartesian Products

- **Let A and B be sets. The Cartesian product of A and B, $A \times B$, is defined by**

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}.$$

  - E.g., $\{a, b\} \times \{1, 2\} = \{(a, 1), (a, 2), (b, 1), (b, 2)\}$
  - For finite A and B, $|A \times B| = |A||B|$
  - The Cartesian product is not commutative: $\neg \forall AB: A \times B = B \times A$.

- **The Cartesian product of the sets $A_1, A_2, \ldots, A_n$ is defined by**

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, \ldots, n\}.$$

- **Given a predicate P and a domain D, the truth set of P is the set of elements $x \in D$ such that $P(x)$ is true. The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.**
  - Q: considering integers, $P(x)$ is "$|x|=1$." What is the truth set of P?
  - A:

Review: Set Notations So Far

- **Variable objects *x*, *y*, *z*; sets *S*, *T*, *U*.**
- **Literal set {*a*, *b*, *c*} and set-builder {*x*|*P*(*x*)}.**
- **$\in$ relational operator**
- **The empty set $\emptyset$.**
- **Set relations: =, $\subseteq$, $\supseteq$, $\subset$, $\supset$, $\not\subset$, etc.**
- **Venn diagrams.**
- **Cardinality |*S*|**
- **Infinite sets N, Z, R.**
- **Power sets P(*S*).**
- **Cartesian products $S \times T$**

Set Operations: Union ($\cup$) & Intersection ($\cap$)

- **Let A and B be sets. The union of A and B, $A \cup B$, is the set containing elements from A or B.**

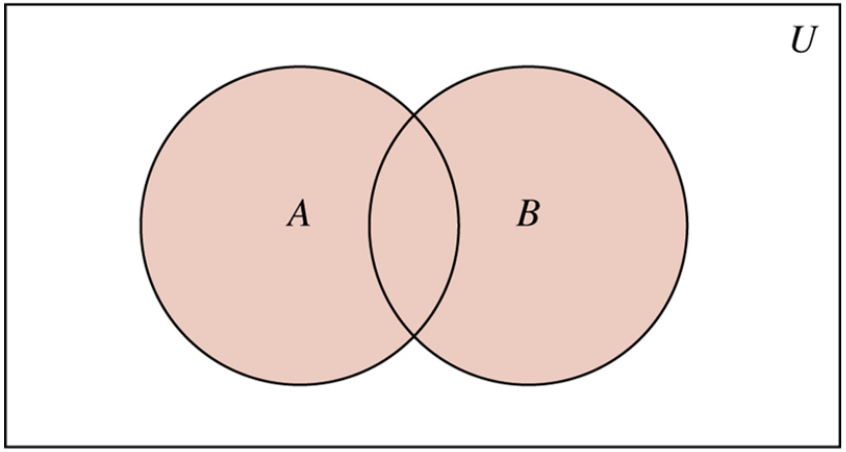
  $$A \cup B = \{x \mid x \in A \vee x \in B\}$$

  - $A \cup B \supseteq A$; $A \cup B \supseteq B$

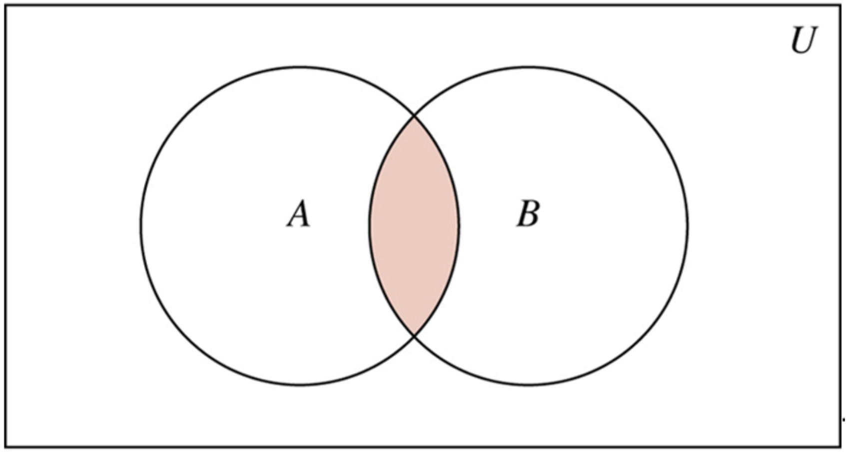
$\cup$nion

- **Let A and B be sets. The intersection of A and B, $A \cap B$, is the set containing elements in both A and B.**

  $$A \cap B = \{x \mid x \in A \wedge x \in B\}$$



$A \cup B$



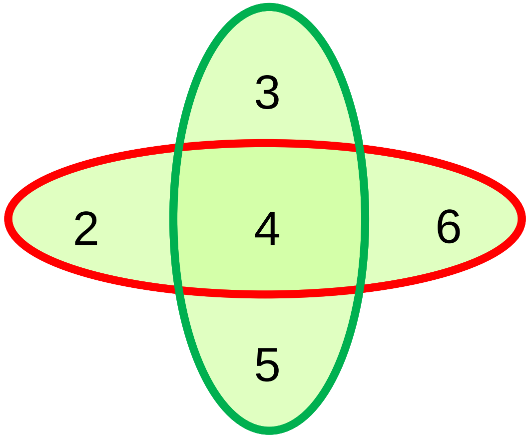
$A \cap B$

Example: Union & Intersection

Union

- $\{a, b, c\} \cup \{2, 3\} = \{a, b, c, 2, 3\}$
- $\{2, 4, 6\} \cup \{3, 4, 5\}$

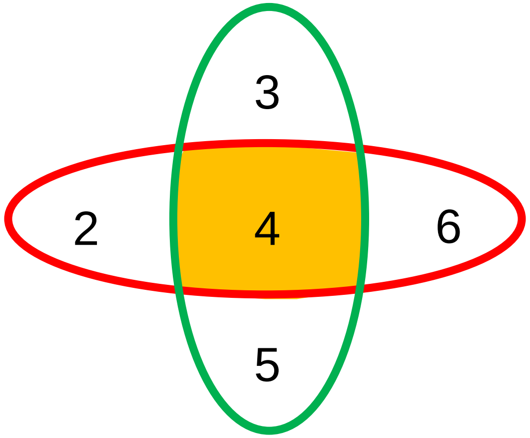
$= \{2, 4, 6, 3, 4, 5\} = \{2, 3, 4, 5, 6\}$



Intersection

- $\{a, b, c\} \cap \{2, 3\} =$
- $\{2, 4, 6\} \cap \{3, 4, 5\}$

$=$

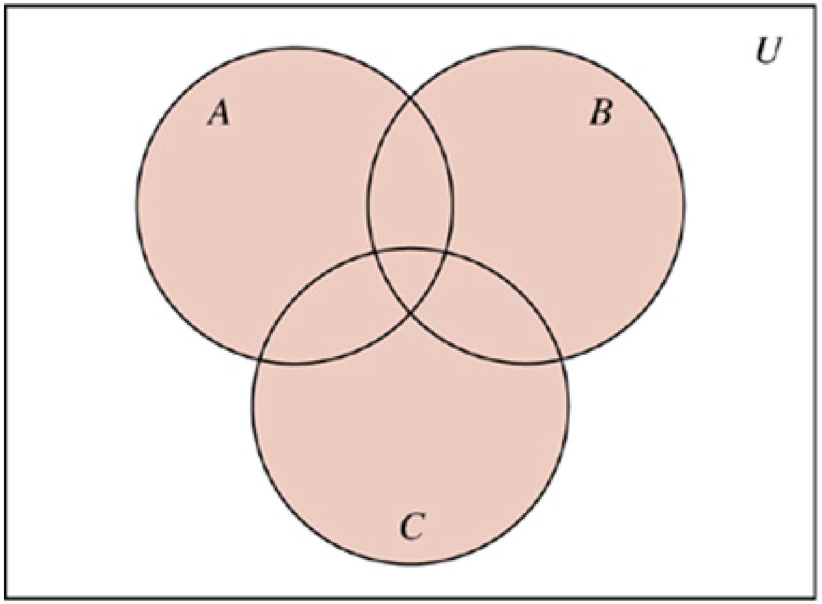


Generalized Union & Intersection

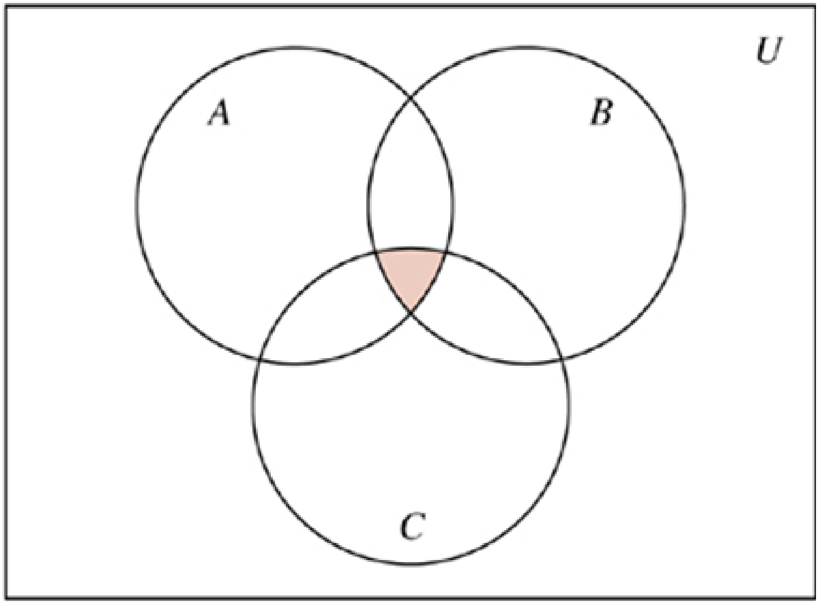
- Let $A_1, A_2, \ldots, A_n$ be sets.
  Define $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$ **and** $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$

- **Remark: Let I be a set (not necessarily finite). Suppose we have a set A_i for each $i \in I$. Then**
  $\bigcup_{i \in I} A_i = \{x \mid x \in A_i$ **for some** $i \in I\}$ **and** $\bigcap_{i \in I} A_i = \{x \mid x \in A_i$ **for all** $i \in I\}$

$A \cup B \cup C$



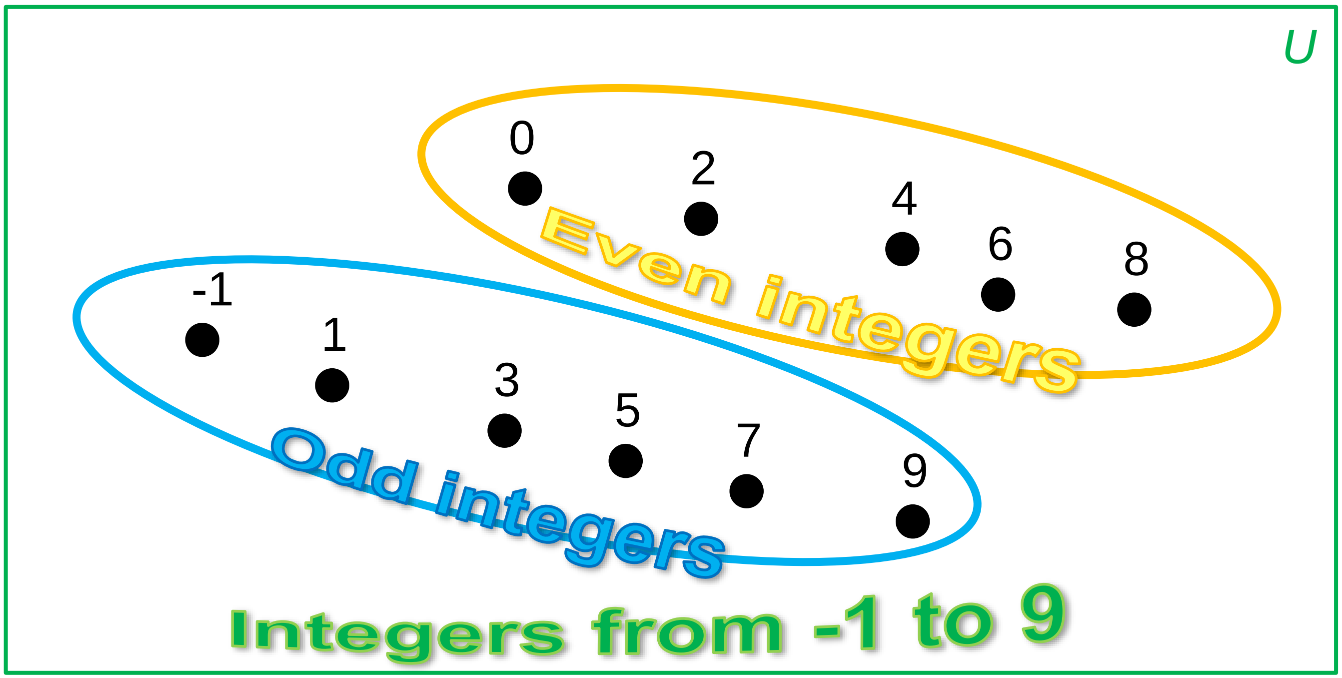
$A \cap B \cap C$



I: index set

Disjoint Sets

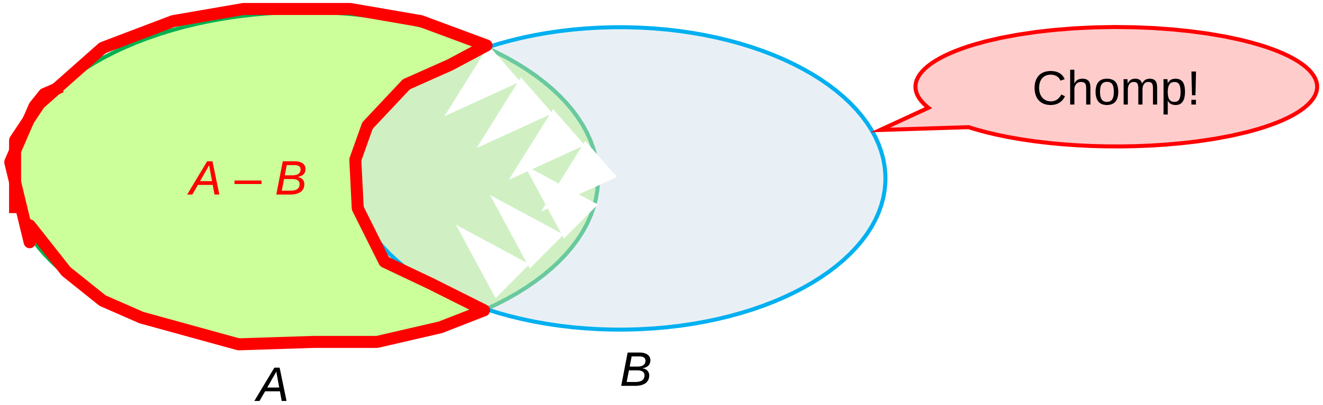
- **Two sets A, B are disjoint if their intersection is the empty set, i.e., $A \cap B = \varnothing$**
- **E.g., the set of even integers is disjoint with the set of odd integers.**



Set Difference

- Let *A* and *B* be sets. The difference of *A* and *B*, *A* – *B*, is the set that contains elements in *A* but not in *B*.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

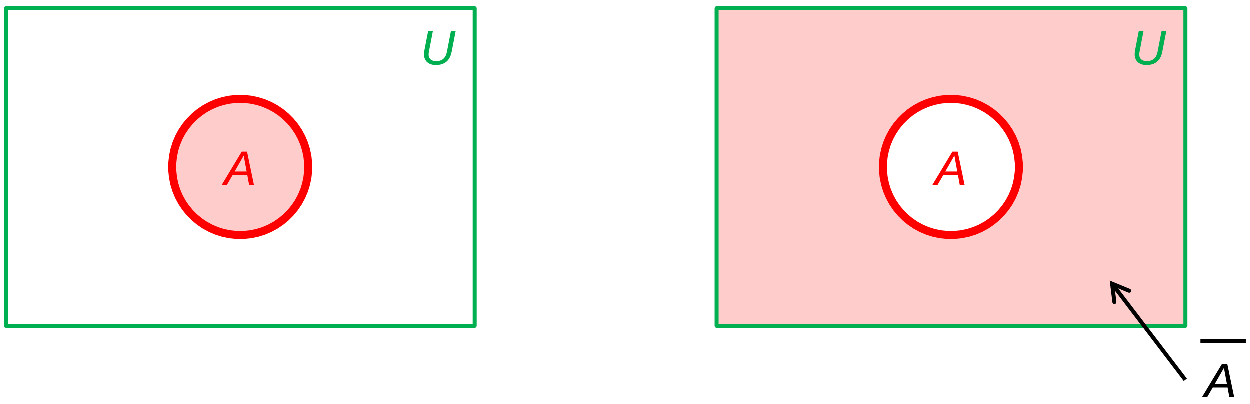


- E.g., {1, 2, 3, 4, 5, 6} – {2, 3, 5, 7, 9, 11} =
- Q: Z – N =?

Set Complement

- Let *U* be the universal set. The **complement** of the set *A*, $\overline{A}$ or A^c, is the complement of *A* with respect to *U*.

$$\overline{A} = U - A$$



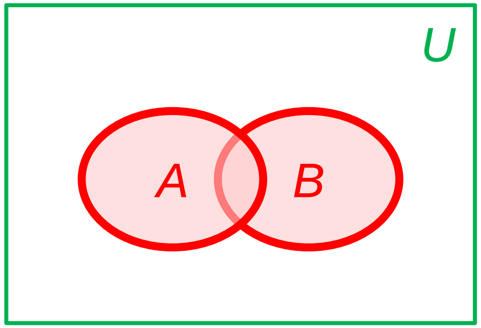
- E.g., let *U* be *N*. $\overline{\{3,5\}} = \{0,1,2,4,6,7,\ldots\}$

Set Identities (1/2)

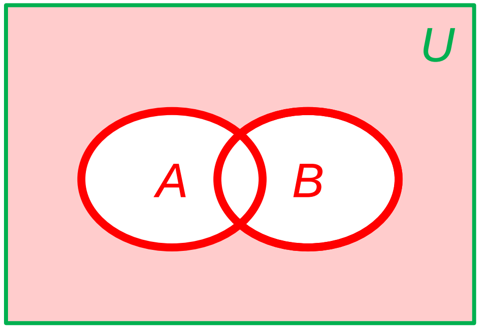
Identity	Name
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\bar{\bar{A}} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws

Set Identities (2/2)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \bar{A} = U$ $A \cap \bar{A} = \emptyset$	Complement laws



$A \cup B$



$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Example

- **Let *A*, *B*, and *C* be sets. Show $\overline{A\cup(B\cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$.**
- **Pf:**
  - $\overline{A\cup(B\cap C)} = \overline{A} \cap \overline{(B\cap C)}$
    $= \overline{A} \cap (\overline{B} \cup \overline{C})$
    $= (\overline{B} \cup \overline{C}) \cap \overline{A}$
    $= (\overline{C} \cup \overline{B}) \cap \overline{A}$

Proving Set Identities

- **To prove statements about sets of the form $E_1 = E_2$ (where E_s are set expressions), here are three useful techniques:**

1. **Mutual subsets: prove $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$ separately.**
2. **Use set builder notation & logical equivalences.**
3. **Use a membership table.**

Method 1: Mutual subsets

- **Example: Show $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$.**
- **Pf:**
  1. Show $A\cap(B\cup C) \subseteq (A\cap B)\cup(A\cap C)$.
     - Assume $x\in A\cap(B\cup C)$, & show $x\in(A\cap B)\cup(A\cap C)$.
     - We know that $x\in A$, and either $x\in B$ or $x\in C$.
       - Case 1: $x\in B$. Then $x\in A\cap B$, so $x\in(A\cap B)\cup(A\cap C)$.
       - Case 2: $x\in C$. Then $x\in A\cap C$, so $x\in(A\cap B)\cup(A\cap C)$.
     - Therefore, $x\in(A\cap B)\cup(A\cap C)$.
     - Therefore, $A\cap(B\cup C)\subseteq(A\cap B)\cup(A\cap C)$.
  2. Show $(A\cap B)\cup(A\cap C) \subseteq A\cap(B\cup C)$. …

Method 2: Set Builder Notation & Logical Equivalences

- $\mathbf{A\cap(B\cup C)} = \{\, x \mid x\in A \wedge x\in(B\cup C) \,\}$

  $= \{\, x \mid x\in A \wedge (x\in B \vee x\in C) \,\}$

  $= \{\, x \mid (x\in A \wedge x\in B) \vee (x\in A \wedge x\in C) \,\}$

  $= \{\, x \mid x\in(A\cap B\,) \vee x\in(A\cap C) \,\}$

  $= \{\, x \mid x\in(A\cap B)\cup(A\cap C) \,\}$

  $= (A\cap B)\cup(A\cap C)$

Method 3: Membership Tables

- **Membership table**
  - Just like truth tables for propositional logic.
  - Columns for different set expressions.
  - Rows for all combinations of memberships in constituent sets.
  - Use “1” to indicate membership in the derived set, “0” for non-membership.
  - Prove equivalence with identical columns.
- **E.g., show $(A \cup B) - B = A - B$.**

A	B	$A \cup B$	$(A \cup B) - B$	$A - B$
0	0	0	0	0
0	1	1	0	0
1	0	1	1	1
1	1	1	0	0

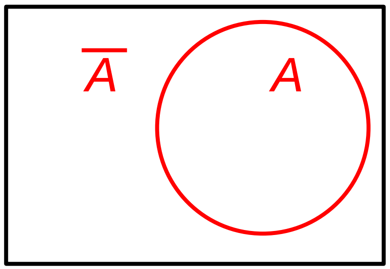
Representing Sets with Bit Strings

- **How to represent sets using a computer?**

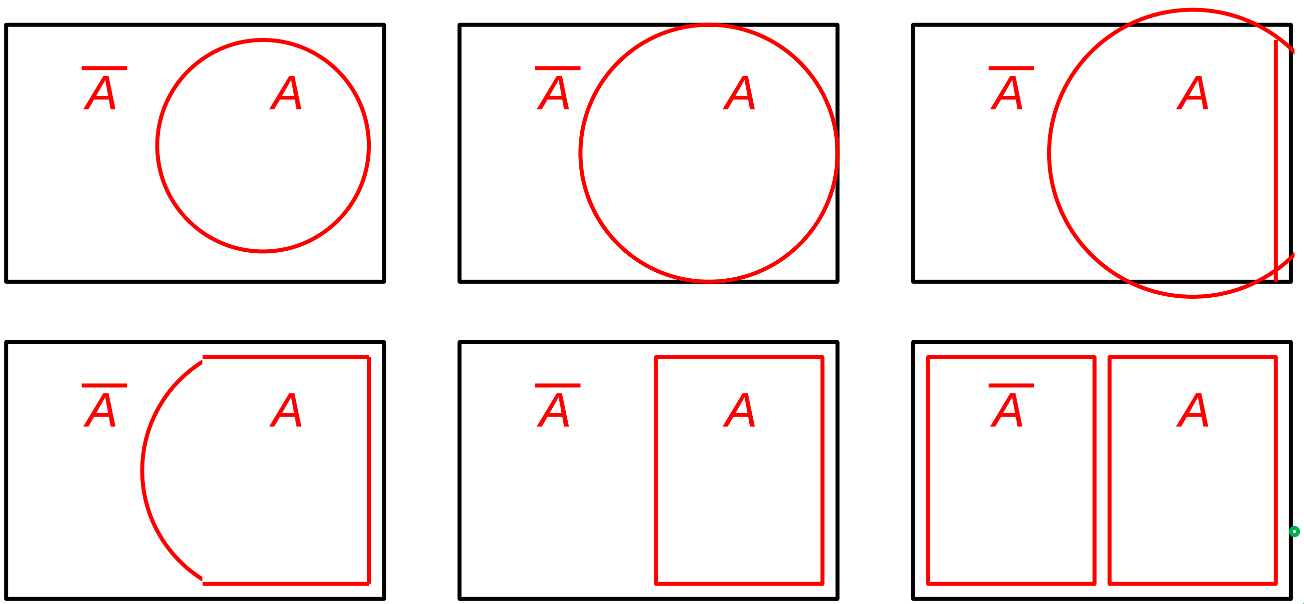
1. **Store the elements of a set in an unordered fashion.**
   - The operations of computing the union, intersection, or difference of two sets would be time-consuming, due to a large amount of searching for elements.
2. **Store elements using an arbitrary ordering of the elements of the universal set.**
   - Specify an arbitrary ordering of the elements of U, $\{x_1, x_2, \ldots, x_n\}$, represent a finite set $A \subseteq U$ as the bit string of length n, $b_1 b_2 \ldots b_n$, where the i^{th} bit in this string is 1 if x_i belongs to A and is 0 otherwise.
   - E.g. $U=\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, A contains odd integers in U.
   - A = 1010101010.
   - Q: How to represent even integers?
   - Q: How to do union, intersection, difference operations?

From Venn diagram to Karnaugh map (1/2)

- **Start with circle *A* in a universal set**

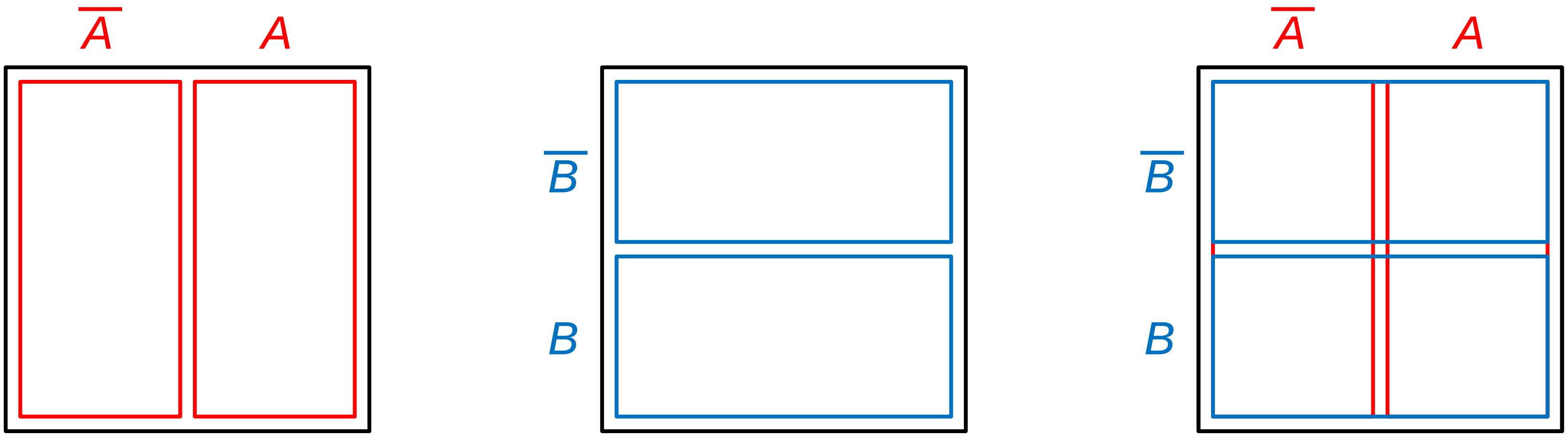


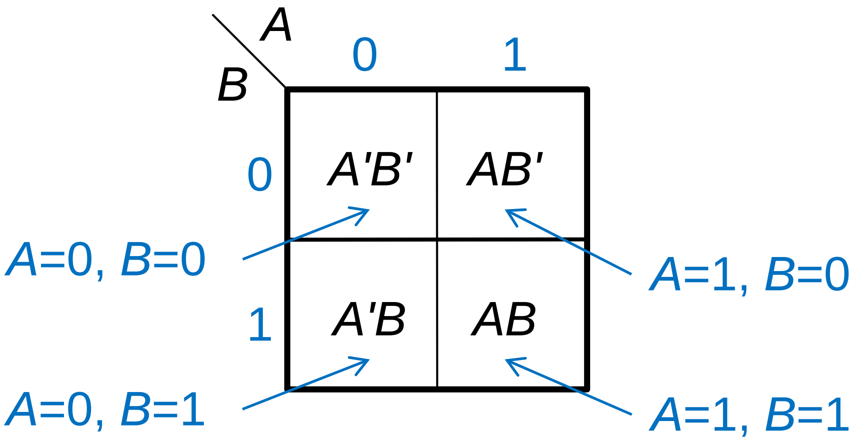
- **Morph a Venn diagram into almost a Karnaugh map**



Tony R. Kuphaldt, Lessons in Electric Circuits, 2007

From Venn diagram to Karnaugh map (2/2)





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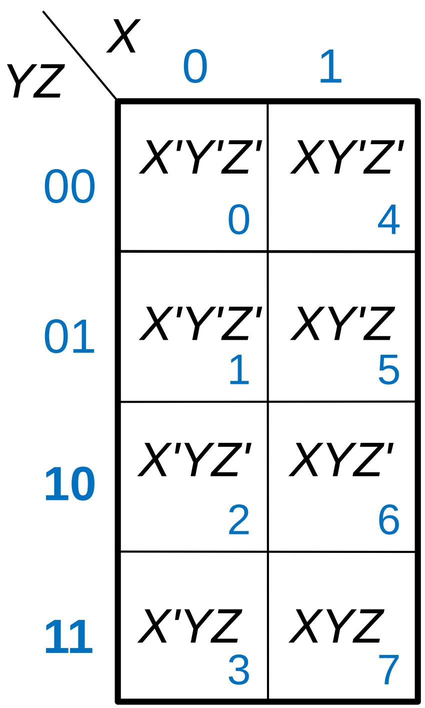
Veitch Diagram vs. Karnaugh Map

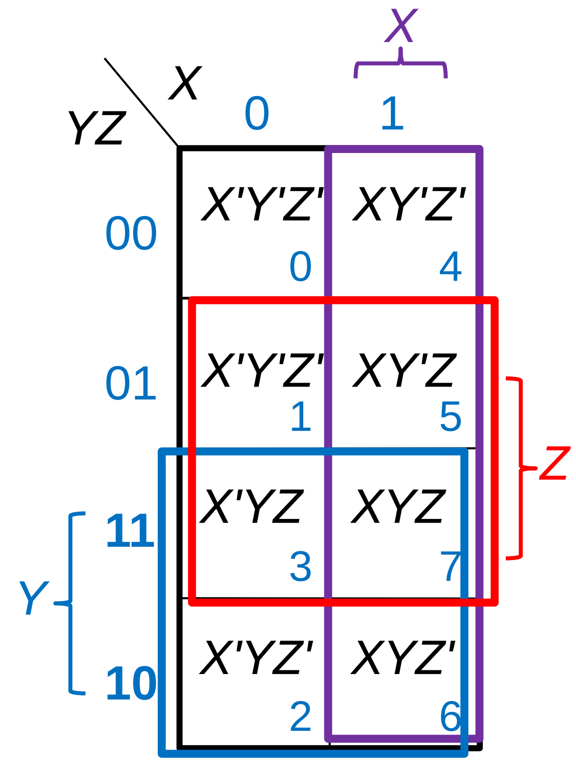
Veith diagram, 1952

- Edward W. Veitch (1924)

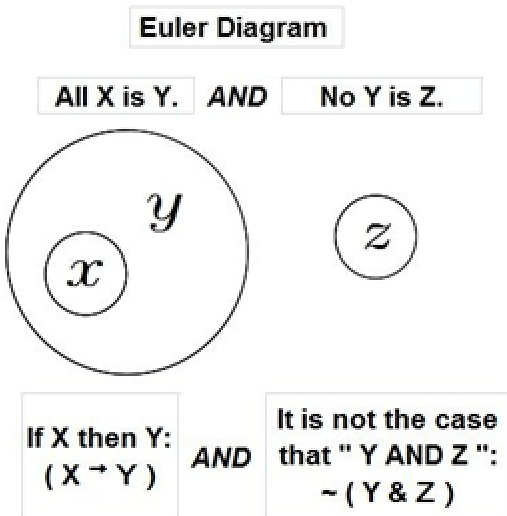


Karnaugh map, 1953

- Maurice Karnaugh (1924)
- Refine Veith diagram



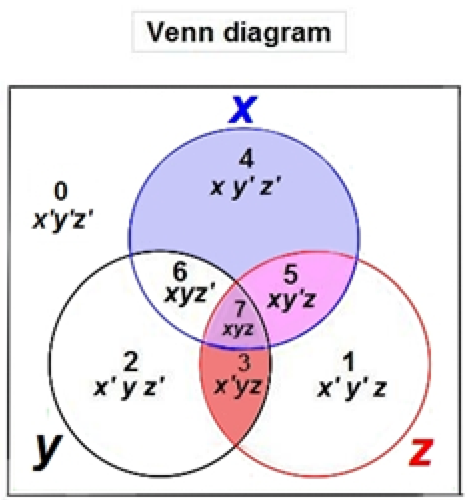
Euler- to Venn-diagram and Karnaugh map

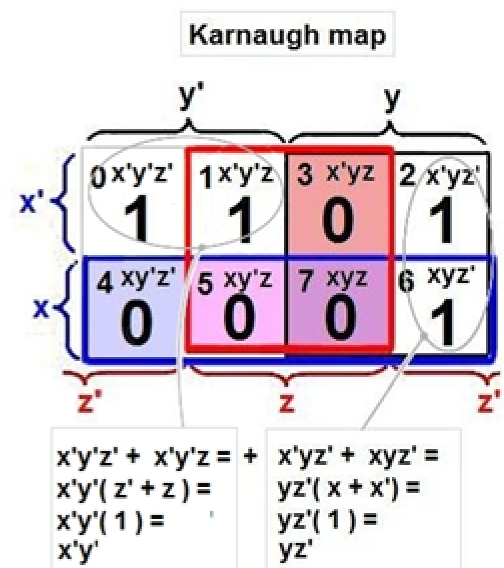


Truth table

Truth table output

truth table / Venn / Karnaugh	row # / area # / square #	x	y	z	(~	(y	&	z)	&	(x	→	y))
		variable			NOT (Y & Z)					If an X then a Y		
x'y'z'	0	0	0	0	1	0	0	0	1	0	1	0
x'y'z	1	0	0	1	1	0	0	1	1	0	1	0
x'yz'	2	0	1	0	1	1	0	0	1	0	1	1
x'yz	3	0	1	1	0	1	1	1	0	0	1	1
xy'z'	4	1	0	0	1	0	0	0	0	1	0	0
xy'z	5	1	0	1	1	0	0	1	0	1	0	0
xyz'	6	1	1	0	1	1	0	0	1	1	1	1
xyz	7	1	1	1	0	1	1	1	0	1	1	1





http://en.wikipedia.org/wiki/Euler_diagram

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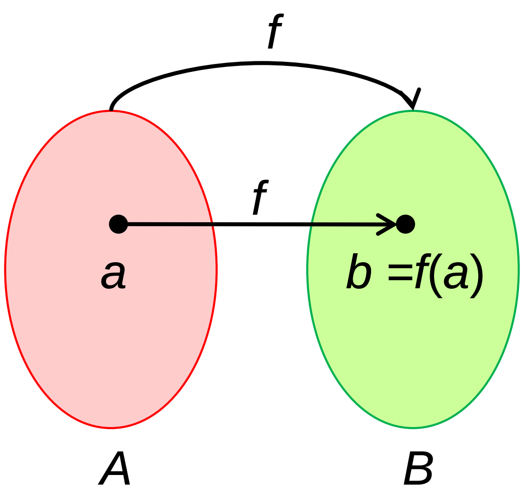
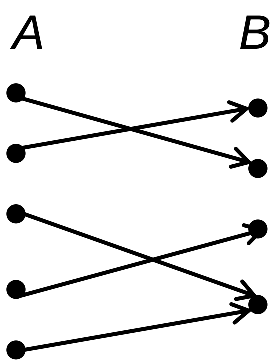
Functions

Function

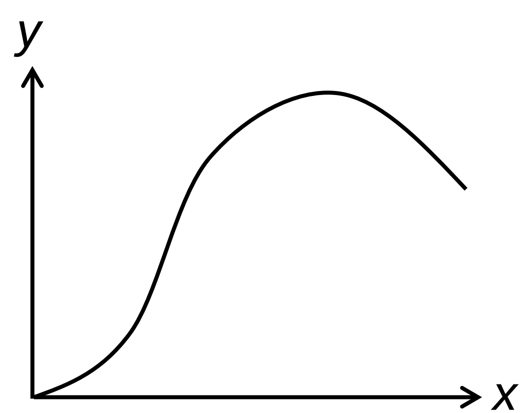
- **Let A and B be sets. A function f from A to B, $f : A \to B$, is an assignment of exactly one element of B to each element of A. Sometimes, we say f maps A to B as well.**
  - mapping, transformation
- **Graphical representations**



Like Venn diagram



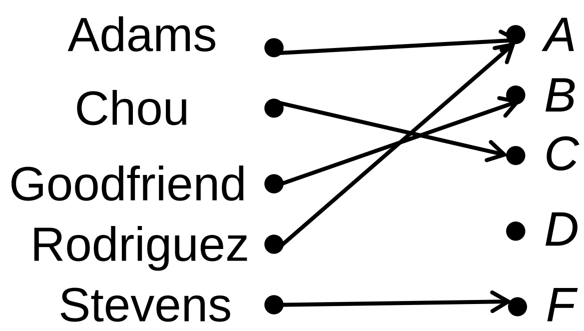
Bipartite Graph



Plot

Example

- Grades of Discrete Mathematics course



Example

- **A proposition can be viewed as a function from "situations" to truth values {T, F}**
  - A logic system called situation theory.
  - *p*="It is raining."; *s*=our situation here, now
  - $p(s) \in \{\mathbf{T}, \mathbf{F}\}$.
- **A propositional operator can be viewed as a function from ordered pairs of truth values to truth values**
  - e.g., $\vee((F, T)) = T$.
  - e.g., $\rightarrow((F, T)) = F$.

Example

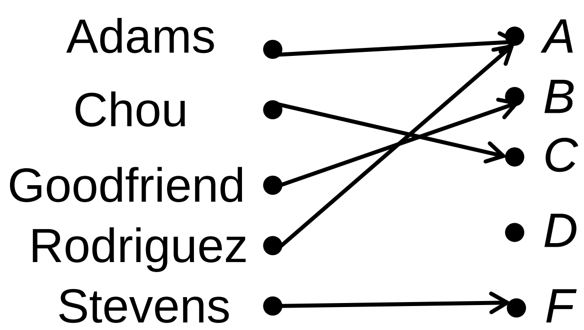
- **A predicate can be viewed as a function from "objects" to truth values {T, F}**
  - e.g., $P(x)$ = "x is 7 feet tall"
  - P(Mike) = "Mike is 7 feet tall" = **F**
- **A bit string *B* of length *n* can be viewed as a function from the numbers {1, …, *n*} (bit positions) to the bits {0, 1}**
  - e.g., $B = 101$, $B(3) = 1$

Example

- **A set *S* over universe *U* can be viewed as a function from the elements of *U* to {T, F}, saying for each element of *U* whether it is in *S*.**
  - e.g., $S = \{3\}$; $S(0) = F$, $S(3) = T$
- **A set operator such as ∩ ∪ — can be viewed as a function from pairs of sets to sets**
  - E.g., $\cap(\{1, 3\}, \{3, 4\}) = \{3\}$

Terminology

- **Let A and B be sets, $f : A \rightarrow B$. $f(a) = b$, where $a \in A$ and $b \in B$**
  - A is the domain of f.
  - B is the codomain of f.
  - b is the image of a under f.
  - a is a preimage of b under f.
    - In general, b may have more than 1 preimage.
  - The range $R \subseteq B$ of f is $\{b \mid \exists a \in A, f(a)=b\}$.
  - The image of $S \subseteq A$, $f(S) = \{b \mid \exists a \in S, f(a)=b\}$.
- **E.g.,**
  - Given the grades of students in DM. Domain? Codomain? Image of Chou? Preimage of A? Range?



Real-Valued Functions

- **Let f and g be functions from A to R. Then $f+g$, fg, $f \circ g$ are also functions from A to R.**

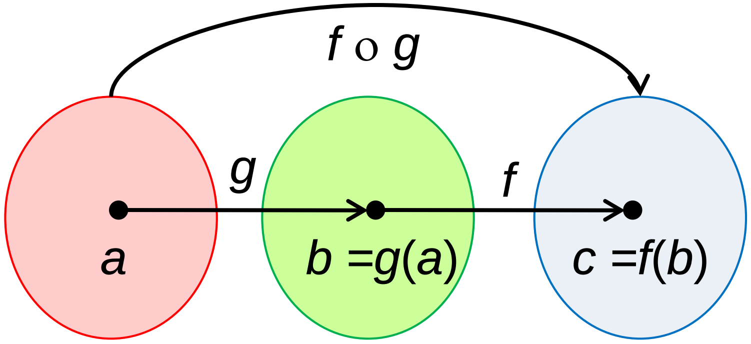
  plus compose

  $$(f+g)(x) = f(x) + g(x)$$
  $$(fg)(x) = f(x)g(x)$$
  $$(f \circ g)(x) = f(g(x))$$

- **E.g.,**

  $(f \circ g)(x)$: composition

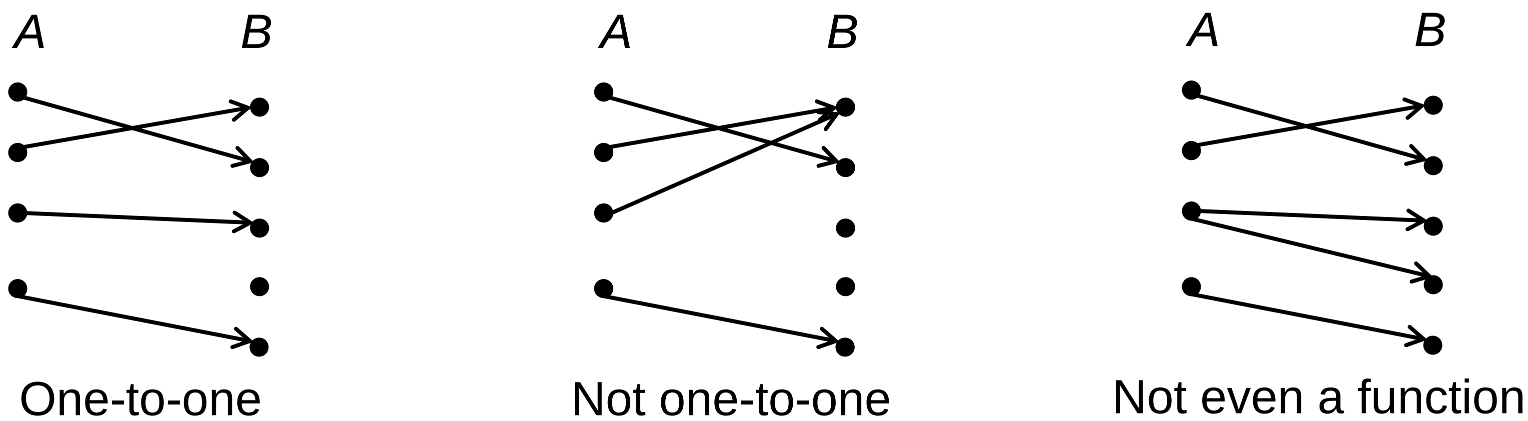
  - Let f and g be functions from **R** to **R** s.t. $f(x) = x^2$ and $g(x) = x - x^2$.
  - $(f + g)(x) = f(x) + g(x) = x$
  - $(fg)(x) = f(x)g(x) = x^2(x - x^2) = x^3 - x^4$
  - $(f \circ g)(x) = f(g(x)) = (x - x^2)^2$



- **A function f whose domain and codomain are subsets of R is called strictly increasing if $f(x) < f(y)$ whenever $x < y$. f is called strictly decreasing if $f(x) > f(y)$ whenever $x < y$.**

One-to-One Functions

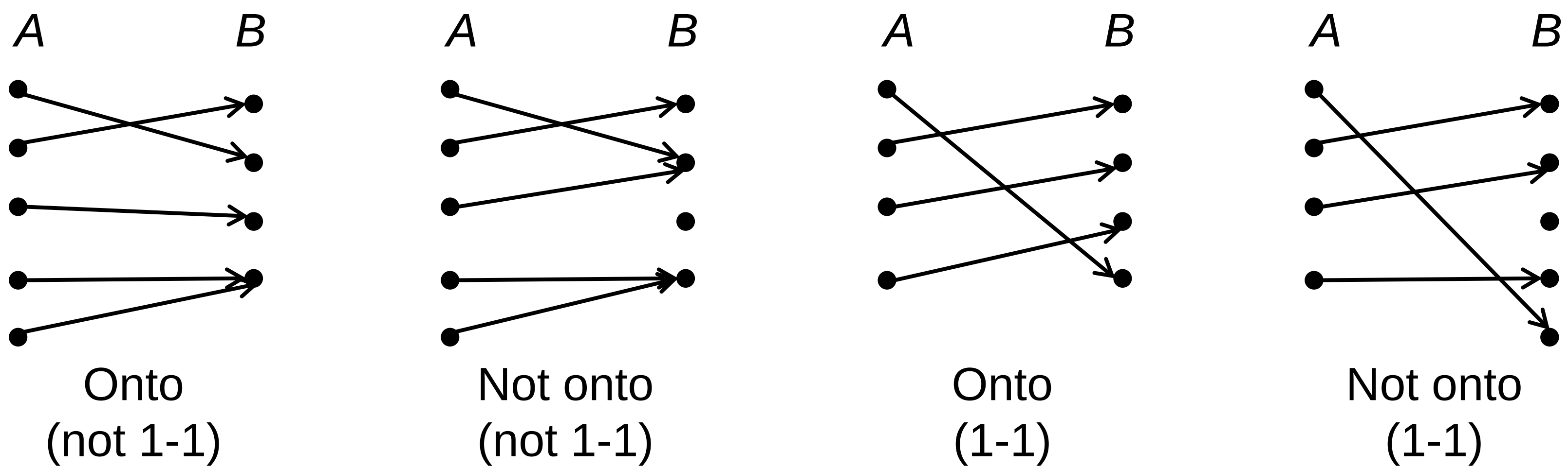
- **A function $f : A \to B$ is said to be one-to-one, or injective, if and only if $f(x) = f(y)$ implies $x = y$ for all x and y in the domain.**
  - $\forall x \forall y\ ((f(x) = f(y)) \leftrightarrow (x = y))$
  - $\forall x \forall y\ ((f(x) \neq f(y)) \leftrightarrow (x \neq y))$
  - Domain and range have the same cardinality



- **Q: Is $f(x) = x^2$ a one-to-one function where $x \in$ R?**
- **Q: Is $f(x) = x^2$ a one-to-one function where $x \in$ N?**
- **Q: Is $f(x) = x+1$ a one-to-one function where $x \in$ R?**

Onto Functions

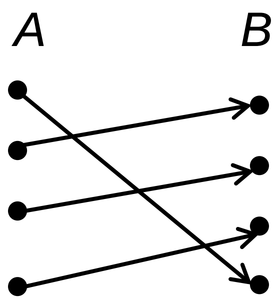
- A function $f : A \to B$ is said to be **onto**, or **surjective**, if and only if for any $b \in B$ there is an $a \in A$ such that $f(a) = b$.
  - Range is equal to its codomain



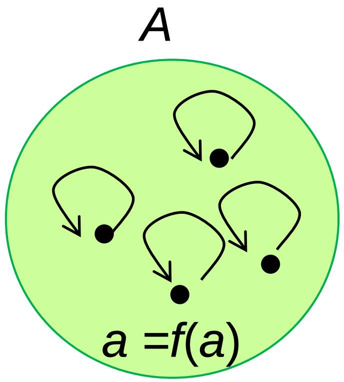
- **Q: Is $f(x) = x^2$ an onto function where $x \in R$?**
- **Q: Is $f(x) = x^3$ an onto function where $x \in R$?**
- **Q: Is $f(x) = x+1$ an onto function where $x \in Z$?**

Bijection

- **A function *f* is a one-to-one correspondence, or a bijection, if it is both injective and surjective.**



Bijection:
One-to-one and Onto



domain and range

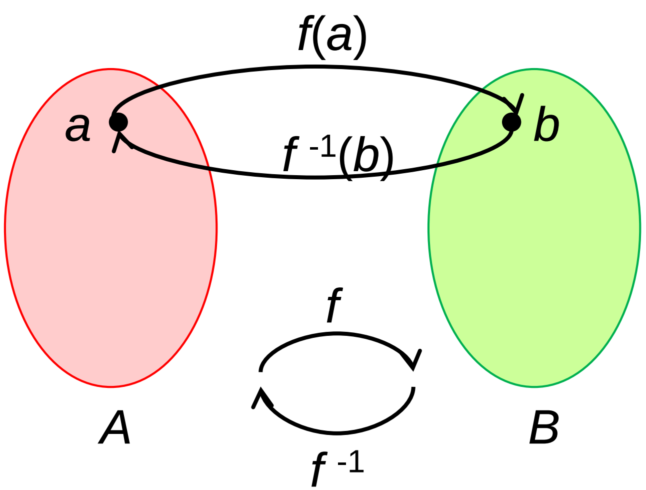
- **E.g., identity functions are bijections.**
  - For any domain A, The identity function $I_A: A \to A$, where $I_A(x) = x$ for all $x \in A$.
  - Some identity functions:

+ing 0, *ing by 1, ∧ing with **T**, ∨ing with **F**, ∪ing with ∅, ∩ing with U

Inverse (1/2)

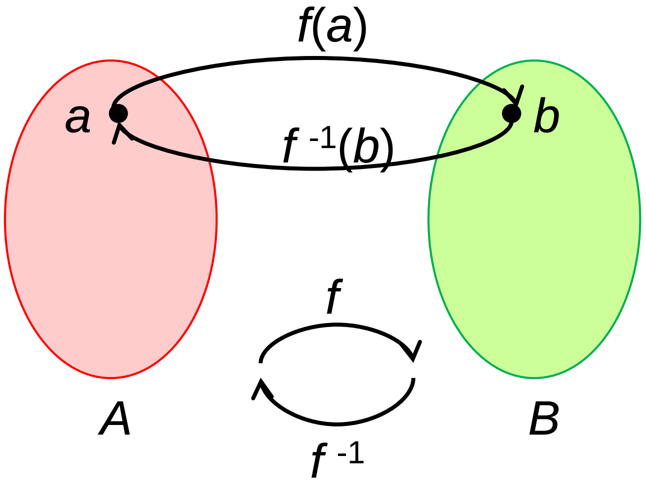
- **Let $f : A \to B$ be a bijection. The inverse function of f, f^{-1}, is a function from B to A such that $f^{-1}(b) = a$ when $f(a) = b$.**

- **Q: Is f^{-1} well-defined? That is, is it a function?**
- **Q: Is $f(x) = x^2$ invertible where $x \in R$?**
- **Q: Is $f(x) = x^3$ invertible where $x \in R$?**
- **Q: Is $f(x) = x+1$ invertible where $x \in Z$?**



Inverse (2/2)

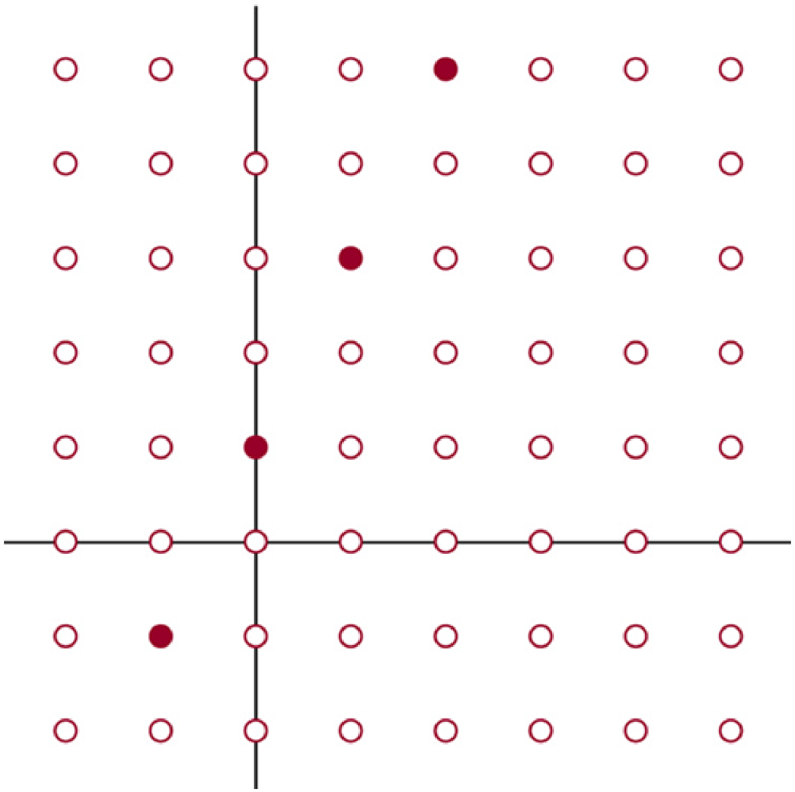
- **Given a bijection *f* and its inverse f^{-1}, $f(a) = b \rightarrow f^{-1}(b) = a$**
  - Q: $(f^{-1} \circ f)(a) = ?$
  - Q: $(f \circ f^{-1})(b) = ?$



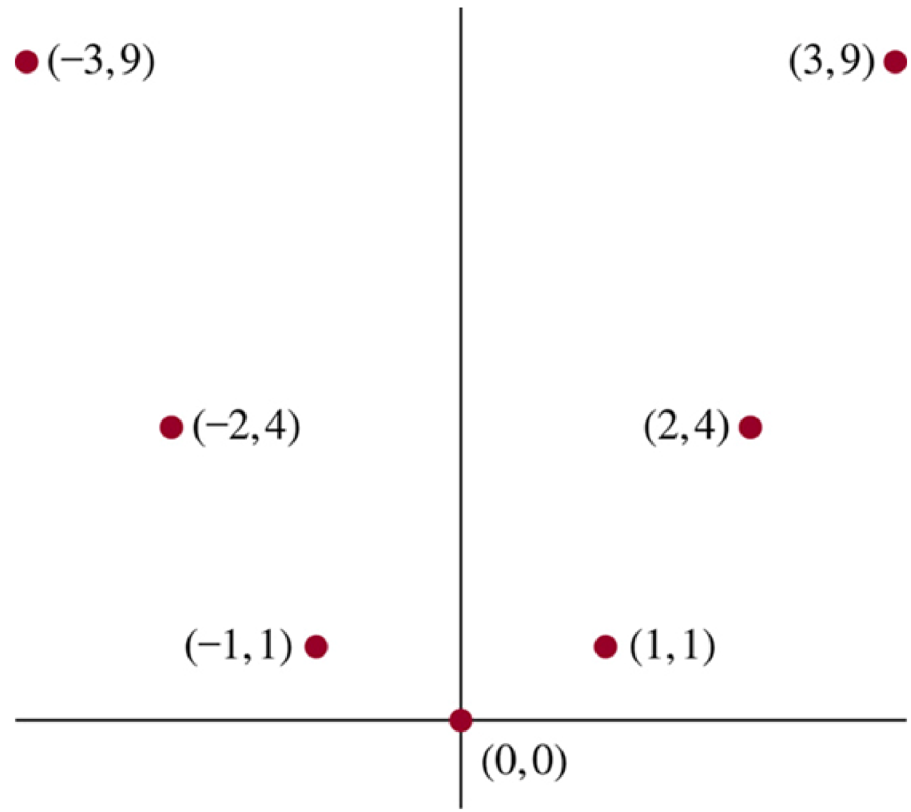
Graph

- **Let $f : A \rightarrow B$. The graph of f is the set (ordered pairs)**

  $\{(a, b) \mid a \in A \text{ and } f(a)=b\}$.



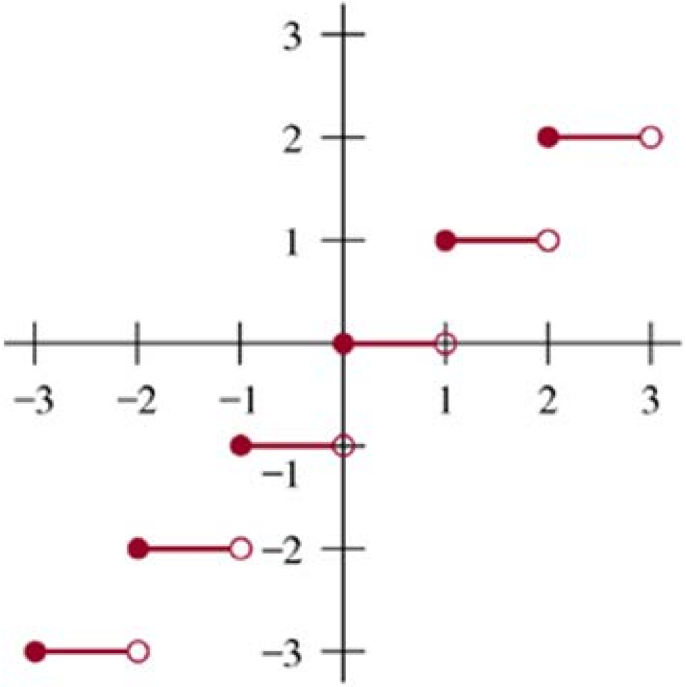
The graph of $f(n) = 2n+1$ from **Z** to **Z**



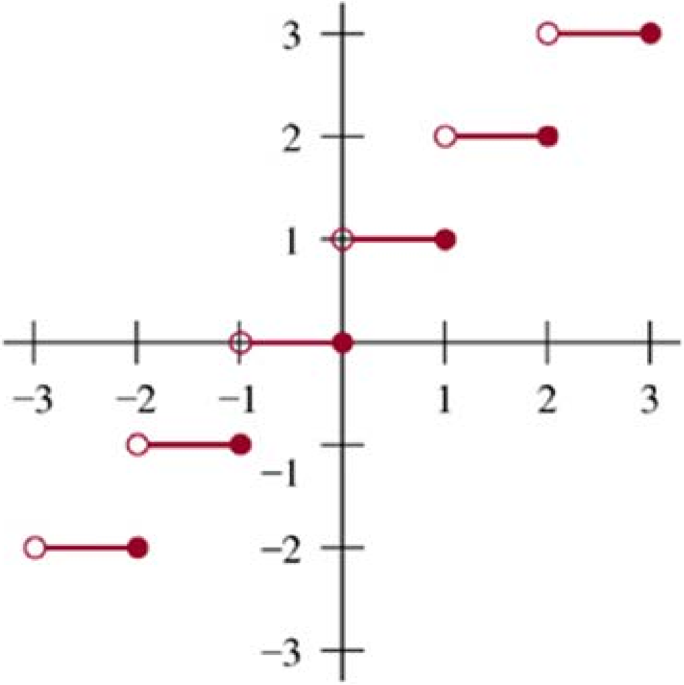
The graph of $f(x) = x^2$ from **Z** to **Z**

Floor and Ceiling Functions

- The **floor** function $\lfloor x \rfloor$ assigns the largest integer that is less than or equal to x.
- The **ceiling** function $\lceil x \rceil$ assigns the smallest integer that is greater than or equal to x.



$y = \lfloor x \rfloor$



$y = \lceil x \rceil$

Example

- **Let $x \in$ R. Show $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x+½ \rfloor$.**
- **Pf.**
  - Let $x = n + \varepsilon$, where $n \in \mathbf{Z}$ and $0 \le \varepsilon < 1$. Then $n = \lfloor x \rfloor$.
  - Consider the following two cases:
  - $0 \le \varepsilon < ½$.
    - Hence, $\lfloor 2x \rfloor = \lfloor 2n + 2\varepsilon \rfloor = 2n$.
    - On the other hand, $\lfloor x \rfloor + \lfloor x+½ \rfloor = 2n$.
  - $½ \le \varepsilon < 1$.
    - Hence, $\lfloor 2x \rfloor = \lfloor 2n + 2\varepsilon \rfloor = 2n+1$.
    - On the other hand, $\lfloor x \rfloor + \lfloor x+½ \rfloor = 2n+1$.

Properties of Floor/Ceiling Functions

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x+1$
(2)	$x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x+n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x+n \rceil = \lceil x \rceil + n$

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Sequences

Sequence

- A **sequence** is a function from Z to a set *S*. We use a_n to denote the image of the integer *n* and call a_n a **term** of the sequence.

- E.g., consider a sequence $\{a_n\}$, where $a_n = 1/n$.
- The list of the terms of this sequence

  $a_1, a_2, a_3, \ldots$

  starts with

  $1, 1/2, 1/3, \ldots$

Geometric Progression

- **A geometric progression is a sequence of the form**

  $$a, ar, ar^2, \ldots, ar^n, \ldots$$

  where the initial term a and the common ratio r are in R.
- **E.g.,**
  - $\{b_n\} = (-1)^n$
  - $\{c_n\} = 2*5^n$
  - $\{d_n\} = 6*(1/3)^n$

Arithmetic Progression

- **A arithmetic progression is a sequence of the form**

  $$a, a+d, a+2d, \ldots, a+nd, \ldots$$

  where the initial term *a* and the common difference *d* are in R.
- **E.g.,**
  - $\{s_n\} = -1 + 4n$
  - $\{t_n\} = 7 - 3n$

Special Integer Sequences

- **Q: Given a few terms, find formulae for the sequences**
  - 1, 1/2, 1/4, 1/8, …
  - 1, 3, 5, 7, …
  - 1, -1, 1, -1, …
  - 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, …
  - 5, 11, 17, 23, 29, …
  - 1, 7, 25, 79, 241, …

TABLE 1 **Some Useful Sequences.**

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, … .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, … .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, … .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, … .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, … .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, … .

Summations

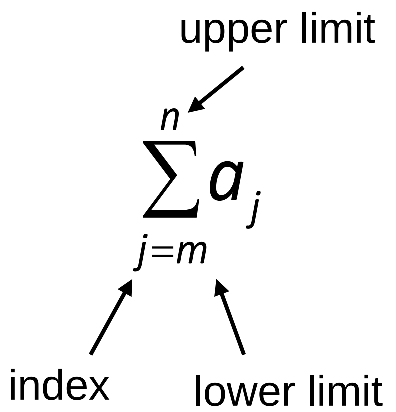
- **Consider the following summation notation**

$$\sum_{j=m}^{n} a_j = a_m + a_{m+1} + \ldots + a_n$$

where *j* is called the index of summation, *m* its lower limit, and *n* its upper limit.

- **E.g,**

$$\sum_{j=1}^{100} (-1)^j =$$



Summation Manipulations

- **Distributive law** $$\sum_x cf(x) = c\sum_x f(x)$$
- **Application of commutativity** $$\sum_x \big(f(x)+g(x)\big) = \left(\sum_x f(x)\right) + \sum_x g(x)$$
- **Index shifting** $$\sum_{i=j}^{k} f(i) = \sum_{i=j+n}^{k+n} f(i-n)$$
- **Series splitting** $$\sum_{i=j}^{k} f(i) = \left(\sum_{i=j}^{m} f(i)\right) + \sum_{i=m+1}^{k} f(i) \quad \text{if } j \le m < k$$
- **Order reversal** $$\sum_{i=j}^{k} f(i) = \sum_{i=0}^{k-j} f(k-i)$$
- **Grouping** $$\sum_{i=0}^{2k+1} f(i) = \sum_{i=0}^{k} f(2i) + f(2i+1)$$

Useful Summation Formulae

Sum	Closed form
$\sum_{k=0}^{n} ar^k (r \neq 0)$	$\frac{ar^{n+1}-a}{r-1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \|x\| < 1$	$\frac{1}{1-x}$
$\sum_{k=0}^{\infty} kx^{k-1}, \|x\| < 1$	$\frac{1}{(1-x)^2}$

Cardinality

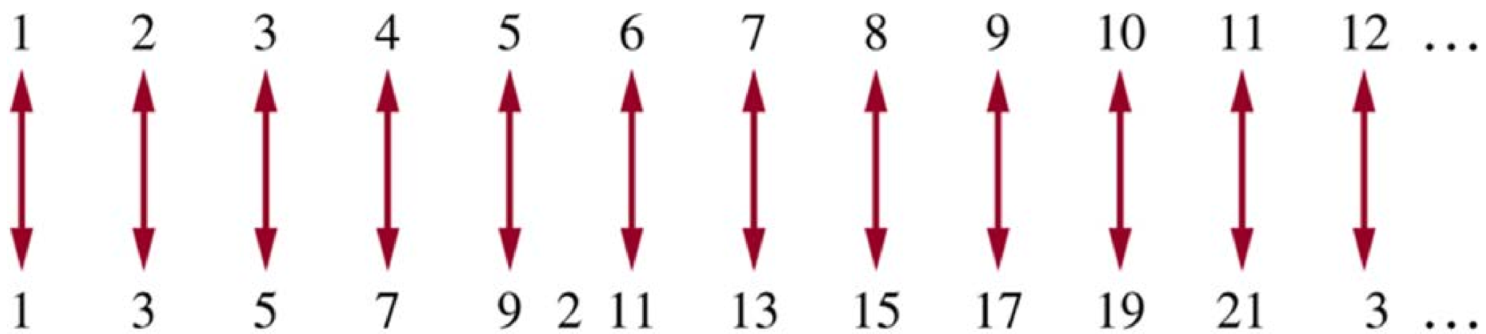
- **For finite sets, cardinality intuitively corresponds to the size of sets. If *A* and *B* are of the same size, we have |*A*| = |*B*|.**

- **How about infinite sets?**
- **A set *A* is countably infinite if there is a bijection *f* : N → *A*. In this case, define the cardinality of *A*, |*A*|, to be N_0. A set *A* is countable if it is finite or countably infinite.**
- **i.e., if there is a bijection from N to *A*, we think N and *A* are of the same size.**
  - For finite sets *A* and *B*, if there is a bijection from *A* to *B*, |*A*| = |*B*|.
  - This corresponds to our intuition of cardinality for finite sets as well.

Example: Countably Infinite Sets

- **Show the following sets are countably infinite: N, Z^+**
- **Pf.** **By definition, find a bijection *f* : N → *A***
  - **N**: $f(n) = n$ is a bijection from **N** to **N**. Thus **N** is countable.
  - **Z^+**: $g(n) = n+1$ is a bijection from **N** to **Z^+**. $|Z^+| = N_0$.

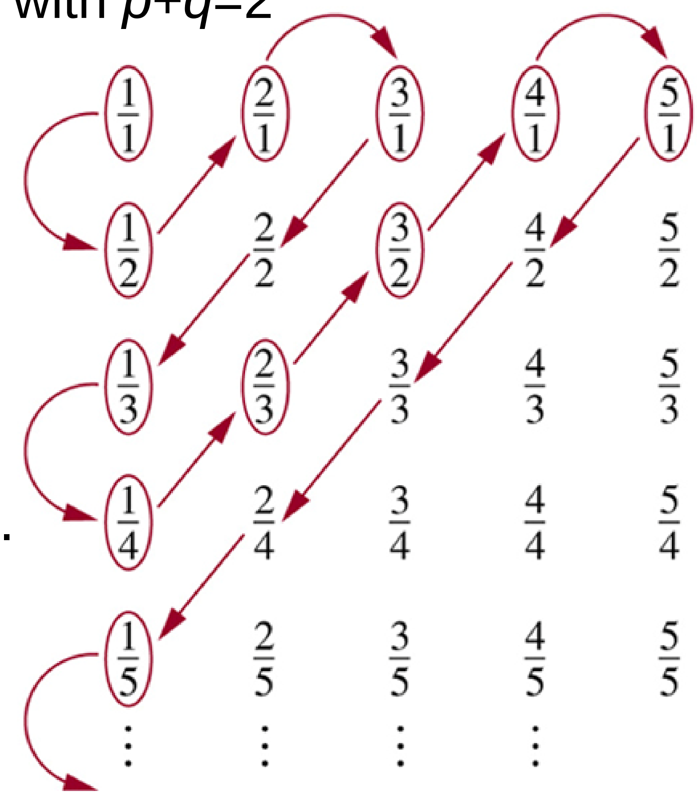
- **Q: odd positive integers?**



- **Q: How about Z?**
- **A: Yes. DIY.**
- **Q: How about NxN?**
- **A: DIY**

Positive Rational Numbers Are Countable

- **Similar to Cantor** **diagonalization** **argument.**
  - We prove something countable instead of uncountable here
- **Key idea:**
  - List the positive rational number *p/q* with *p*+*q*=2
  - List *p*+*q*=3 …
  - Take k = 2,3,4,...
    and for each value of k list all the fractions *p*/*q* with $p + q = k$ and gcd(*p*, *q*)=1, in acsending order of *p*.



http://www.homeschoolmath.net/teaching/rational-numbers-countable.php
http://www.mathpages.com/home/kmath371.htm

Summary on Countability

- **For any two (possibly infinite) sets *A* and *B*, |*A*| = |*B*| iff there is a bijection *f* : *A* → *B*.**

- **A set *A* is countable if there is an injective function *f* : *A* → N.**

- **A set *A* is countably infinite if there is a bijection *f* : N → *A*. In this case, define the cardinality of *A*, |*A*|, to be N_0. A set *A* is countable if it is finite or countably infinite.**

Russell's Paradox (1/2)

- **Consider the following statement: “The Serbian barber only shaves those who do not shave themselves." Now ask yourself: does the barber shave himself or not?**
  - There are only two cases: either he shaves himself, or he doesn't.
  - Suppose he shaves himself. We are told that he does not shave those who shave themselves. Hence he does not shave himself.
  - Now suppose he does not shave himself. We are told that he shaves those who do not shave themselves. Hence he does shave himself.
  - Both cases lead to contradiction. What's going on here?

- **The barber paradox is an intriguing question raised by philosophers. It seems like a tricky game of words. And nobody expects it would have anything to do with mathematics. However, Russell is able to exploit the idea and create a similar paradox in mathematics in 1903.**

Russell's Paradox (2/2)

- **Consider the set $A = \{x \mid x \notin x\}$. Since $\varnothing \notin \varnothing$, we have $\varnothing \in A$. A does seem to make sense. Now, can you tell me whether $A \in A$?**
  - Again, there are only two possibilities: either $A \in A$ or $A \notin A$.
  - Suppose $A \in A$. Since any element x of A has the property that $x \notin x$, in particular $A \notin A$. A contradiction.
  - On the other hand, suppose $A \notin A$. Then by the definition of A, $A \in A$. Another contradiction.
- **The arguments of Russell's paradox are similar to those in barber's paradox. In both cases, we cannot tell the truth value of a proposition. In philosophy, it may be a game of language. But it is a serious matter in the foundation of mathematics.**
- **Mathematicians now distinguish small from large sets. Mathematics is still good if we pay close attention to the collection of all sets (thus the name large set). In fact, Russell's paradox can be avoided if we do not allow the collection of all sets as the universe.**

Don't worry in this class