CHAPTER 7
ADVANCED COUNTING

Outline

- **Content**
  - Recurrence relations
  - Solving linear recurrence relations
  - Divide-and-conquer algorithms and recurrence relations
  - Generating functions
  - Inclusion-exclusion
- **Reading**
  - Chapter 7

Most of the following slides are by courtesy of Prof. J.-D. Huang and Prof. H.-C. Yen

Recurrence Relations

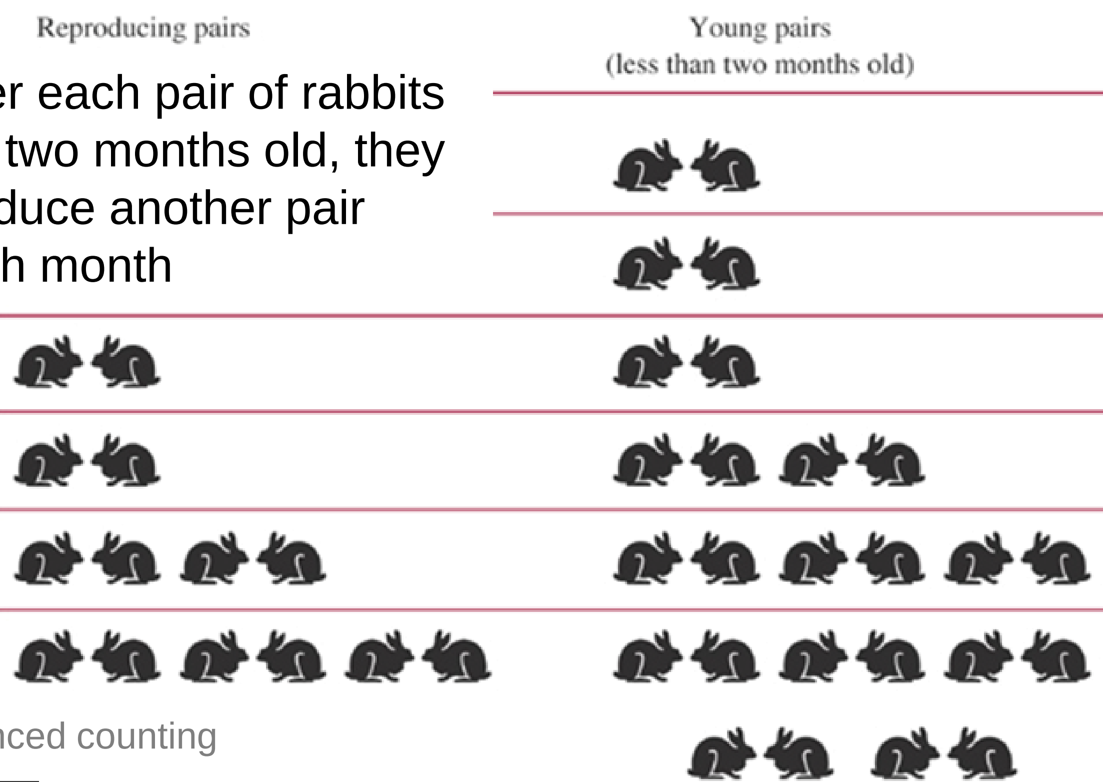
- **A recurrence relation for the $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, $a_0, a_1, \ldots, a_{n-1}$.**

- **A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.**

Rabbits and Fibonacci Numbers

- **E.g., the relation $f_n = f_{n-1} + f_{n-2}$ of Fibonacci numbers is a recurrence relation.**
  - A sequence $\{f_n\}$:
  - Initial condition: $f_1 = 1$, $f_2 = 1$
  - Recurrence: $f_n = f_{n-1} + f_{n-2}$, for $n \geq 3$

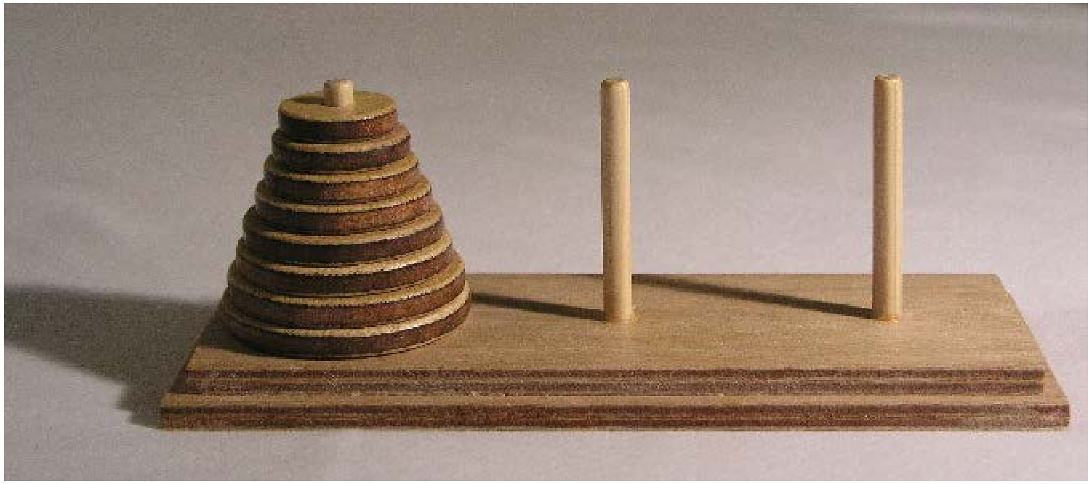
After each pair of rabbits are two months old, they produce another pair each month



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

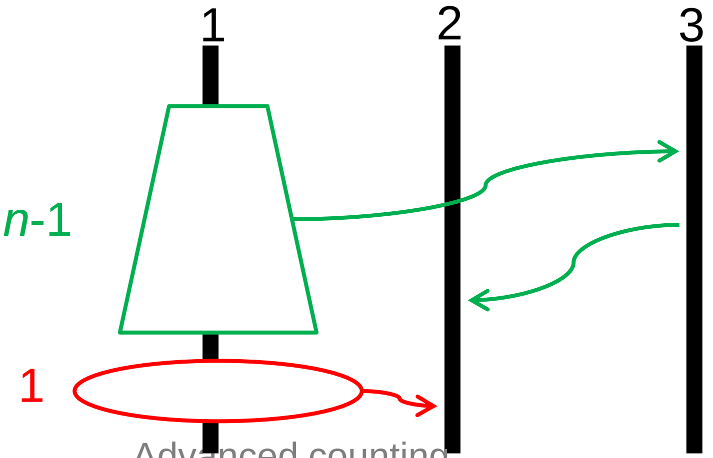
Tower of Hanoi (1/3)

- **Consider moving a stack of disks with different sizes with three pegs. Initially, all disks are sorted and placed on the first peg.**
  - Only move 1 disk at a time (topmost).
  - Never set a larger disk on a smaller one (always sorted).



Tower of Hanoi (2/3)

- **Sol:**
  - Let H_n be the number of steps to solve the problem of n disks.
  - Clearly, $H_1 = 1$.
  - Now consider moving n disks from peg 1 to peg 2.
    - If we can move the topmost n -1 disks from peg 1 to peg 3, we can solve the puzzle by moving the bottom disk from peg 1 to peg 2 and then the n - 1 disks from peg 3 to peg 2. In other words, $H_n = H_{n-1} + 1 + H_{n-1}$.
  - We have $H_n = 2H_{n-1} + 1$, for $n \geq 2$.



http://en.wikipedia.org/wiki/File:Tower_of_Hanoi_4.gif

Tower of Hanoi (3/3)

- $H_n = 2H_{n-1} + 1$
  $= 2(2H_{n-2} + 1) + 1$
  $= 2^2 H_{n-2} + 2^1 + 2^0$
  $= 2^2(2H_{n-3} + 1) + 2^1 + 2^0$
  $= 2^3 H_{n-3} + 2^2 + 2^1 + 2^0$
  …
  $= 2^{n-1} H_1 + 2^{n-2} + \ldots + 2^1 + 2^0$, $H_1 = 1$
  $= \sum_{i=0..n-1} 2^i$
  $= (2^n - 1)/(2-1) = 2^n - 1$
- **A myth says monks are transferring 64 gold disks. The world will end when they finish the job.**
  - If the monks take 1 second to move a disk
  - $2^{64} - 1$ sec = 18,446,744,073,709,551,615 sec > 500 billion years
  - we should sleep well ☺

Bit Strings without 2 Consecutive 0s

- **Find a recurrence relation and give initial conditions for the number of bit strings of length *n* that do not have 2 consecutive 0s**
  - Let a_n denote the number of qualified bit strings of length n

End with a 1 | any bit string of length n-1 with no 2 consecutive 0s | 1 | $\Rightarrow a_{n-1}$

End with a 0 | any bit string of length n-2 with no 2 consecutive 0s | 1 0 | $\Rightarrow a_{n-2}$

  - Recurrence: $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$
  - Initial conditions: $a_1 = 2$, $a_2 = 3$

Catalan Numbers

- **Find the number of ways to parenthesize the product of $n + 1$ numbers $x_0, x_1, \ldots, x_n$.**
- **Sol:**
  - Let C_n denote the # of ways to parenthesize the product of n+1 numbers, e.g., $C_3 = 5$

    $x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$ $\quad$ $x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)$ $\quad$ $(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$ $\quad$ …
  - Clearly, $C_0 = 1$.
  - Consider $x_0, x_1, \ldots, x_n$. We can compute it by $(x_0 \cdot \ldots \cdot x_k) \cdot (x_{k+1} \cdot \ldots \cdot x_n)$ for any $k = 0, \ldots, n$. Therefore

    $$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \ldots + C_{n-1} C_0$$
  - The sequence $\{C_n\}$, $C_n = \sum_{k=0..n-1} C_k C_{n-k-1}$, is called Catalan numbers.

Homogeneous Linear Recurrence Relations with Constant Coefficients

- **A linear homogeneous recurrence relation of degree k with constant coefficients is of the form**

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k},$$

  where $c_1, c_2, \ldots, c_k \in \mathrm{R}$ with $c_k \neq 0$.

- **E.g., the recurrence for Fibonacci numbers $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of degree two.**

What does linear means?

If s_n and t_n are solutions, for any real c and d, $cs_n + dt_n$ is solution, too.

Solving Linear Recurrence Relations (1/2)

- **Let $c_1, c_2 \in$ R. Suppose $x^2 = c_1x + c_2$ has two distinct roots r_1 and r_2. Then $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r_1^n + \alpha_2r_2^n$ for $n \in$ N and some constants α_1, α_2.**
- **Pf: ($\Leftarrow$)**
  - Suppose $a_n = \alpha_1r_1^n + \alpha_2r_2^n$.
  - We want to verify $a_n = c_1a_{n-1} + c_2a_{n-2}$.
  - $c_1a_{n-1} = c_1\alpha_1r_1^{n-1} + c_1\alpha_2r_2^{n-1}$
  - $c_2a_{n-2} = c_2\alpha_1r_1^{n-2} + c_2\alpha_2r_2^{n-2}$
  - $c_1a_{n-1} + c_2a_{n-2} = \alpha_1r_1^{n-2}(c_1r_1 + c_2) + \alpha_2r_2^{n-2}(c_1r_2 + c_2)$

$$= \alpha_1r_1^{n-2}(r_1^2) + \alpha_2r_2^{n-2}(r_2^2)$$
$$= \alpha_1r_1^n + \alpha_2r_2^n$$
$$= a_n$$

Solving Linear Recurrence Relations (2/2)

- **Pf: (⇒)**
  - From the first part of proof, we know $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$.
  - It remains to show $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies initial conditions for some α_1, α_2.
  - The theorem then follows from the uniqueness of solution to linear homogeneous recurrence relation.
  - To see $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies initial conditions for some α_1, α_2. Consider $a_1 = \alpha_1 r_1 + \alpha_2 r_2$ and $a_0 = \alpha_1 + \alpha_2$.
  - This is a linear system of two variables α_1, α_2. The solutions are

$$\alpha_1 = \frac{a_1 - a_0 r_2}{r_1 - r_2}, \quad \text{and} \quad \alpha_2 = \frac{a_0 r_1 - a_1}{r_1 - r_2}.$$

Fibonacci Numbers

$ax^2+bx+c=0$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

- **Recall the recurrence relation for Fibonacci numbers $f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0$ and $f_1 = 1$. Find an explicit formula for the Fibonacci numbers.**
- **Sol:**
  - The solutions to $x^2 = x + 1$ are $\frac{1 \pm \sqrt{5}}{2}$
  - Hence, $$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$ for some α_1, α_2.
  - Solving $0 = \alpha_1 + \alpha_2$, and $1 = \alpha_1(\frac{1+\sqrt{5}}{2}) + \alpha_2(\frac{1-\sqrt{5}}{2})$, we have $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$.
  - Therefore, $$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

Let $c_1, c_2 \in$ R. Suppose $x^2 = c_1x + c_2$ has two distinct roots r_1 and r_2. Then $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n \in$ N and some constants α_1, α_2.

Characteristic Equations with Multiple Roots (1/2)

- **Let c_1, $c_2 \in$ R, $c_2 \neq 0$. Suppose $x^2 = c_1x + c_2$ has only one root r. Then $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1 r^n + \alpha_2 n r^n$ for $n \in$ N and some constants α_1, α_2.**
- **Pf: ($\Leftarrow$)**
  - $2r = c_1$ and $c_1r + 2c_2 = 0$
  - Let $a_n = \alpha_1 r^n + \alpha_2 n r^n$. Then
  - $c_1a_{n-1} = c_1\alpha_1 r^{n-1} + c_1\alpha_2(n-1)r^{n-1}$
  - $c_2a_{n-2} = c_2\alpha_1 r^{n-2} + c_2\alpha_2(n-2)r^{n-2}$
  - $c_1a_{n-1} + c_2a_{n-2} = \alpha_1 r^{n-2}(c_1r + c_2) + \alpha_2 r^{n-2}(c_1(n-1)r + c_2(n-2))$

$$= \alpha_1 r^{n-2}r^2 + \alpha_2 r^{n-2}(c_1nr - c_1r + c_2n - 2c_2)$$

$$= \alpha_1 r^n + \alpha_2 r^{n-2}(c_1nr + c_2n)$$

$$= \alpha_1 r^n + \alpha_2 r^{n-2}n(c_1r + c_2)$$

$$= \alpha_1 r^n + \alpha_2 r^{n-2}n(r^2) = \alpha_1 r^n + \alpha_2 n r^n$$

Characteristic Equations with Multiple Roots (2/2)

- **Let c_1, $c_2 \in$ R, $c_2 \neq 0$. Suppose $x^2 = c_1x + c_2$ has only one root r. Then $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1r^n + \alpha_2nr^n$ for $n \in$ N and some constants α_1, α_2.**
- **Pf: ($\Rightarrow$)**
  - It remains to show there are α_1 and α_2 satisfying the initial conditions.
  - We have $a_0 = \alpha_1$.
  - Therefore

$$\alpha_2 = \frac{a_1 - a_0 r}{r}.$$

Example

- **What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?**
- **Sol:**
  - Since 3 is the multiple root of $x^2 = 6x - 9$,
  - we have $a_n = \alpha_1 3^n + \alpha_2 n3^n$.
  - Moreover, $\alpha_1 = a_0 = 1$ and $\alpha_2 = (6-1*3)/3 = 1$.
  - We have $a_n = 3^n + n3^n$.

Let $c_1, c_2 \in R$, $c_2 \neq 0$. Suppose $x^2 = c_1x + c_2$ has only one root r. Then $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = \alpha_1 r^n + \alpha_2 nr^n$ for $n \in N$ and some constants α_1, α_2.

Characteristic Equations with Distinct Roots

- Let $c_1, c_2, \ldots, c_k \in$ R. Suppose the characteristic equation

$$x^k = c_1 x^{k-1} + \ldots + c_k$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$$

for $n \in$ N and some constants $\alpha_1, \alpha_2, \ldots, \alpha_k$.

Characteristic Equations with Multiple Roots

- **Let $c_1, c_2, \ldots, c_k \in$ R. Suppose the characteristic equation**

$$x^k = c_1 x^{k-1} + c_2 x^{k-2} + \ldots + c_k$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$ respectively such that $m_t \geq 1$ and $m_1+m_2+\ldots+m_t = k$.

Then $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m1-1} n^{m1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2,m2-1} n^{m2-1}) r_2^n \\ & + \ldots \\ & + (\alpha_{t,0} + \alpha_{t,1} n + \ldots + \alpha_{t,mt-1} n^{mt-1}) r_t^n \end{aligned}$$

for $n \in$ N and some constants $\alpha_{i,j}$.

Solving Nonhomogeneous Linear Recurrence Relations

- **A linear nonhomogeneous recurrence relation with constant coefficients is of the form**

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

  where $c_1, c_2, \ldots, c_k \in$ R and F : N $\rightarrow$ Z. The recurrence relation

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

  is called the associated homogeneous recurrence relation.

- **Theorem: Let $\{a_n^{(p)}\}$ be a particular solution of**

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n),$$

  Then every solution is of the form $\{a_n^{(p)}+a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of

  $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

  - $a_n^{(h)}$: homogeneous solution
  - $a_n^{(p)}$: particular solution

Example

- **Find all solutions of $a_n = 3a_{n-1} + 2n$, with $a_1 = 3$.**
- **Sol:**
  - $\{a_n\}$ is of the form $\{a_n^{(p)}+a_n^{(h)}\}$.
  - Let us guess $a_n^{(p)} = cn + d$.
  - Then $cn + d = 3(c(n - 1)+ d) +2n = (3c+2)n + (3d - 3c)$.
  - Solve $c = 3c+2$ and $d = 3d - 3c$.
  - We obtain $c = -1$ and $d = -3/2$
  - Hence, $a_n^{(p)} = -n - 3/2$ is a particular solution.
  - We have $a_n^{(h)} = \alpha 3^n$ $(x - 3=0)$
  - Therefore, $a_n = -n - 3/2 + \alpha 3^n$, where α is a constant
  - $3 = a_1 = -1 - 3/2 + 3\alpha$; $\alpha = 11/6$.
  - Finally, $a_n = -n - 3/2 + (11/6)3^n$

Example

- **Find all solutions of $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.**
- **Sol:**
  - $\{a_n\}$ is of the form $\{a_n^{(p)}+a_n^{(h)}\}$.
  - Let us guess $a_n^{(p)} = c\cdot 7^n$.
  - Then $c\cdot 7^n = 5c\cdot 7^{n-1} - 6c\cdot 7^{n-2} + 7^n$; $49c = 35c - 6c + 49$, $c = 49/20$
  - $a_n^{(p)} = (49/20)\cdot 7^n$ is a particular solution.
  - We have $a_n^{(h)} = \alpha_1 2^n + \alpha_2 3^n$ ($x^2 - 5x + 6 = 0$)
  - Therefore, $a_n = \alpha_1 2^n + \alpha_2 3^n + (49/20)\cdot 7^n$ are all solutions.

Solving Particular Solutions

- **Suppose $\{a_n\}$ satisfies $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k \in$ R and**
  $F(n) = (b_tn^t + b_{t-1}n^{t-1} + \ldots + b_1n + b_0)s^n$, where $s, b_0, b_1, \ldots, b_t \in$ R.
- **When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form**
  $(p_tn^t + p_{t-1}n^{t-1} + \ldots + p_1n + p_0)s^n$.
- **When s is a root of the characteristic equation and its multiplicity is m, there is a particular solution of the form**
  $n^m(p_tn^t + p_{t-1}n^{t-1} + \ldots + p_1n + p_0)s^n$.

Example (1/2)

- **Solve $a_n = a_{n-1} + 3n^2 - 3n + 1$, with $a_0 = 0$.**
- **Solution:** Since $a_n = a_{n-1} + 3n^2 - 3n + 1$, $F(n) = (3n^2 - 3n + 1)1^n$.

  we have $a_n^{(h)} = \alpha \cdot 1^n = \alpha$ as homogeneous solutions.

  we let $a_n^{(p)} = n(p_2 n^2 + p_1 n + p_0)$ as a particular solution.

  Substitute $a_n^{(p)}$ in the recurrence relation, we have

  $$n(p_2 n^2 + p_1 n + p_0) = (n-1)(p_2(n-1)^2 + p_1(n-1) + p_0) + 3n^2 - 3n + 1.$$

  Simplify both sides of the equation and compare their coefficients:

  $$\begin{aligned} p_2 &= p_2 \\ p_1 &= -3p_2 + p_1 + 3 \\ p_0 &= 3p_2 - 2p_1 + p_0 - 3 \\ 0 &= -p_2 + p_1 - p_0 + 1 \end{aligned}$$

Example (2/2)

- **Solve $a_n = a_{n-1} + 3n^2 - 3n + 1$, with $a_0 = 0$.**

(Cont'd)
Solve the linear system and get $p_0 = 0, p_1 = 0, p_2 = 1$. Hence $a_n^{(p)} = n(1 \cdot n^2 + 0 \cdot n + 0) = n^3$ is a particular solution.
Thus, $a_n = n^3 + \alpha$ are all solutions. Since $a_0 = 0$, we have $\alpha = 0$. And we conclude $a_n = n^3$.

Notice that $n^3 - (n-1)^3 = 3n^2 - 3n + 1$. Hence $\sum_{k=1}^{n} 3k^2 - 3k + 1 = n^3$. Alternatively, note that

$$a_n = \sum_{k=1}^{n} 3k^2 - 3k + 1,$$

you can solve it by computing the summation.

Divide-and-Conquer Algorithms

- **Divide a problem into 1 or more smaller instances of the same problem**
- **Conquer the subproblems recursively**
- **Combine the solutions**

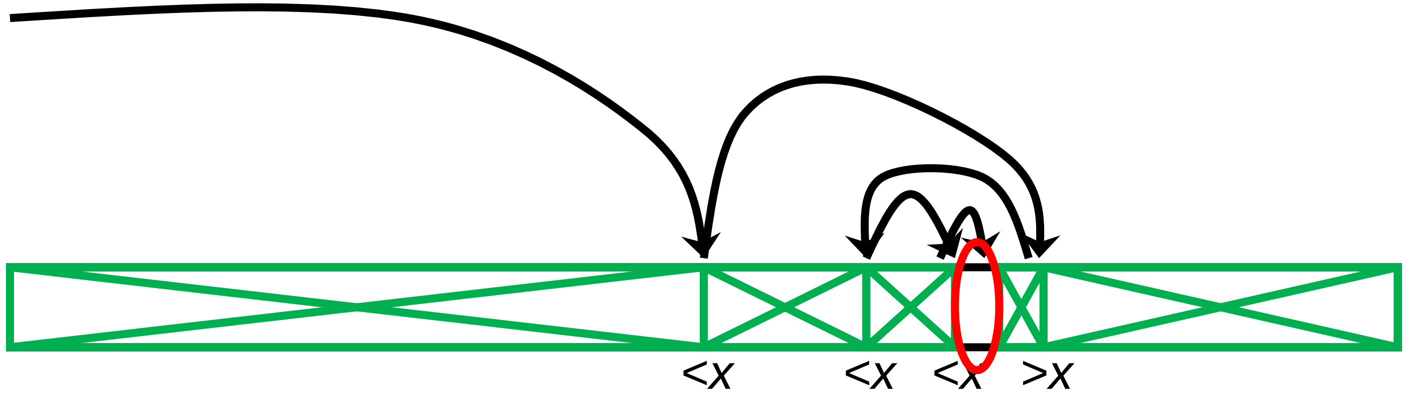
- **Divide-and-conquer algorithms**
  - Binary search
  - Merge sort
  - Tower of Hanoi
  - …

The Recurrence Relation of D&C

- **Assume a divide-and-conquer algorithm**
  - Divides a problem of size n into a subproblems,
    where each subproblem is of size n/b,
    and $g(n)$ extra operations are required in the divide and combine steps
- **Then, if $f(n)$ represents the number of operations required to solve the problem of size n, we obtain the recurrence relation of the form**
  - $f(n) = a\, f(n/b) + g(n)$

Binary Search

- Binary search: Explore the feature of the sorted sequence



ALGORITHM 5 A Recursive Binary Search Algorithm.

procedure *binary search*(x, i, j)
$m := \lfloor (i + j)/2 \rfloor$
if $x = a_m$ **then**
    location $:= m$
else if $(x < a_m$ and $i < m)$ **then**
    binary search$(x, i, m - 1)$
else if $(x > a_m$ and $j > m)$ **then**
    binary search$(x, m + 1, j)$
else *location* $:= 0$

$f(n) = a\, f(n/b) + g(n)$
$\Rightarrow f(n) = f(n/2) + O(1)$

The Recurrence Relation of D&C

- Let f be a non-decreasing function satisfying

  $f(n) = a\,f(n/b) + c$

  whenever $b|n$, where $a \geq 1$, $b \in Z^+$, $b > 1$, and $c \in R^+$. Then

$$f(n) = \begin{cases} O(n^{\log_b a}) & \text{if } a > 1 \\ O(\lg n) & \text{if } a = 1 \end{cases}$$

  Furthermore, when $n = b^k$ and $a > 1$, $k \in Z^+$,

$$f(n) = C_1 n^{\log_b a} + C_2,$$

  where $C_1 = c/(a-1) + f(1)$ and $C_2 = -c/(a-1)$.

- Q: Binary search?
- A: O(lg n)

The Recurrence Relation of D&C

- **Proof:** We first consider $n = b^k$. Then

$$
\begin{aligned}
f(n) &= af(\frac{b^k}{b}) + c \\
&= af(b^{k-1}) + c \\
&= a(af(b^{k-2}) + c) + c \\
&= a^2 f(b^{k-2}) + ac + c \\
&= \cdots \\
&= a^k f(1) + a^{k-1}c + a^{k-2}c + \cdots + ac + c \\
&= a^k f(1) + c\sum_{i=0}^{k-1} a^i
\end{aligned}
$$

When $a = 1$, we have $f(n) = f(1) + ck = f(1) + c\log_b n$. Furthermore, if $b^k < n < b^{k+1}$, we have $f(b^k) \leq f(n) \leq f(b^{k+1})$ and $b^{k+1} < nb$. Thus, $f(n) \leq f(b^{k+1}) = f(1) + c(k+1) < f(1) + c(1 + \log_b n) = f(1) + c + c\log_b n$. Therefore $f(n) = O(\lg n)$ when $a = 1$.

The Recurrence Relation of D&C

(Cont'd)

When $a > 1$ and $n = b^k$, then $n^{\log_b a} = b^{k \log_b a} = a^k$. We have

$$\begin{aligned} f(n) &= a^k f(1) + c\frac{a^k - 1}{a - 1} \\ &= a^k \left[f(1) + \frac{c}{a-1} \right] - \frac{c}{a-1} \\ &= n^{\log_b a} C_1 + C_2 \end{aligned}$$

For $b^k < n < b^{k+1}$, we have

$$\begin{aligned} f(n) &\leq f(b^{k+1}) \\ &= C_1 a^{k+1} + C_2 \\ &= aC_1 a^k + C_2 \\ &\leq aC_1 n^{\log_b a} + C_2 \end{aligned}$$

Hence, $f(n) = O(n^{\log_b a})$.

Example

$$f(n) = \begin{cases} O(n^{\log_b a}) & \text{if } a > 1 \\ O(\lg n) & \text{if } a = 1 \end{cases}$$

- **Let $f(n) = 5f(n/2) + 3$. Estimate $f(n)$.**
- **Sol:**
  - By the above theorem, we have $f(n) = O(n^{\log_2 5}) = O(n^{\lg 5})$
- **For merge sort algorithm, we have $f(n) = 2f(n/2) + \Theta(n)$. The above theorem is not applicable.**

Master Theorem

- Let f be a non-decreasing function satisfying

  $$f(n) = a\, f(n/b) + cn^d$$

  whenever $n=b^k$, where $b, k \in Z^+$ with $a \geq 1$, $b>1$, and $c,d \in R$ with $d \geq 0$, $c>0$. Then

  $$f(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \lg n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

- Q: merge sort: $T(n) = 2\,T(n/2) + \Theta(n)$
- A: $T(n) = O(n \lg n)$

Master Theorem

- Let $a \geq 1$ and $b > 1$ be constants, $f(n)$ be a function, and $T(n)$ be defined on nonnegative integers as

$$T(n) = aT(n/b) + f(n).$$

- Then, $T(n)$ can be bounded asymptotically as follows:

1. $T(n) = \Theta(n^{\log_b a})$ if $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$.
2. $T(n) = \Theta(n^{\log_b a} \lg n)$ if $f(n) = \Theta(n^{\log_b a})$.
3. $T(n) = \Theta(f(n))$ if $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.

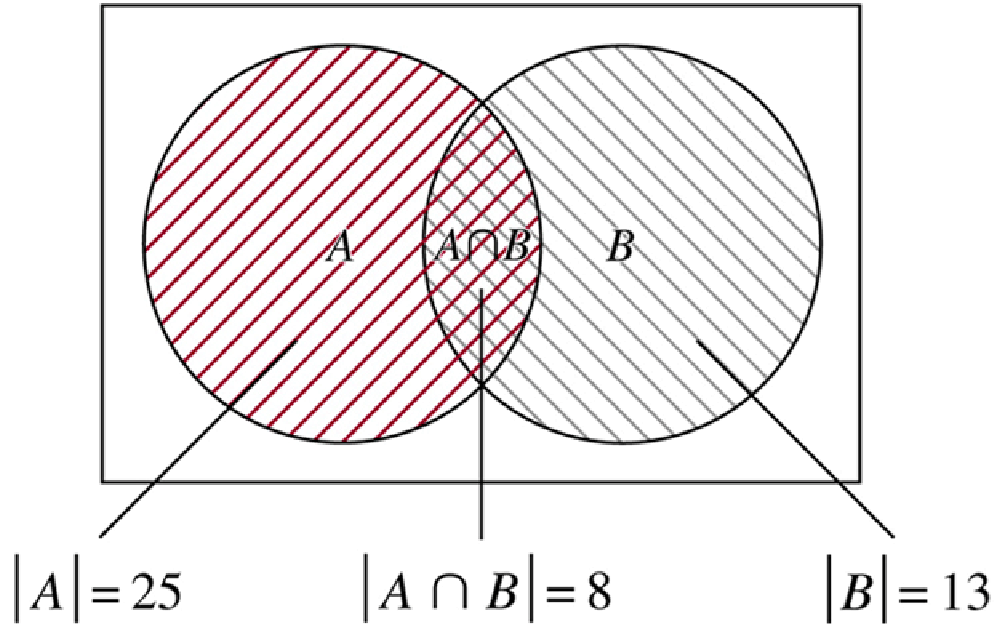
- Intuition: compare $f(n)$ with $\Theta(n^{\log_b a})$.
  - Case 1: $f(n)$ is **polynomially smaller than** $\Theta(n^{\log_b a})$.
  - Case 2: $f(n)$ is **asymptotically equal to** $\Theta(n^{\log_b a})$.
  - Case 3: $f(n)$ is **polynomially larger than** $\Theta(n^{\log_b a})$.

Cormen *et al.*, *Introduction to Algorithms*, 3rd ed., MIT Press, 2009.

Inclusion-Exclusion: Counting for 2 Sets

- $|A \cup B| = |A| + |B| - |A \cap B|$
- **A class contains**
  - 25 students majoring in EE
  - 13 students majoring in CS
  - 8 joint EE and CS majors
- **How many students in this class?**
  - = 25 + 13 - 8 = 30

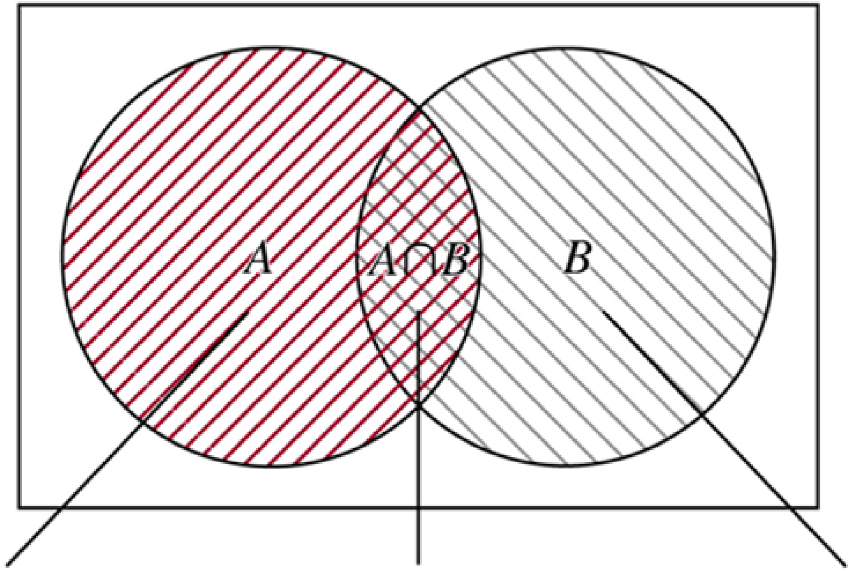
$$|A \cup B| = |A| + |B| - |A \cap B| = 25 + 13 - 8 = 30$$



Example

- **How many positive integers not exceeding 1000 are divisible by 7 or 11?**

$$|A \cup B| = |A| + |B| - |A \cap B| = 142 + 90 - 12 = 220$$

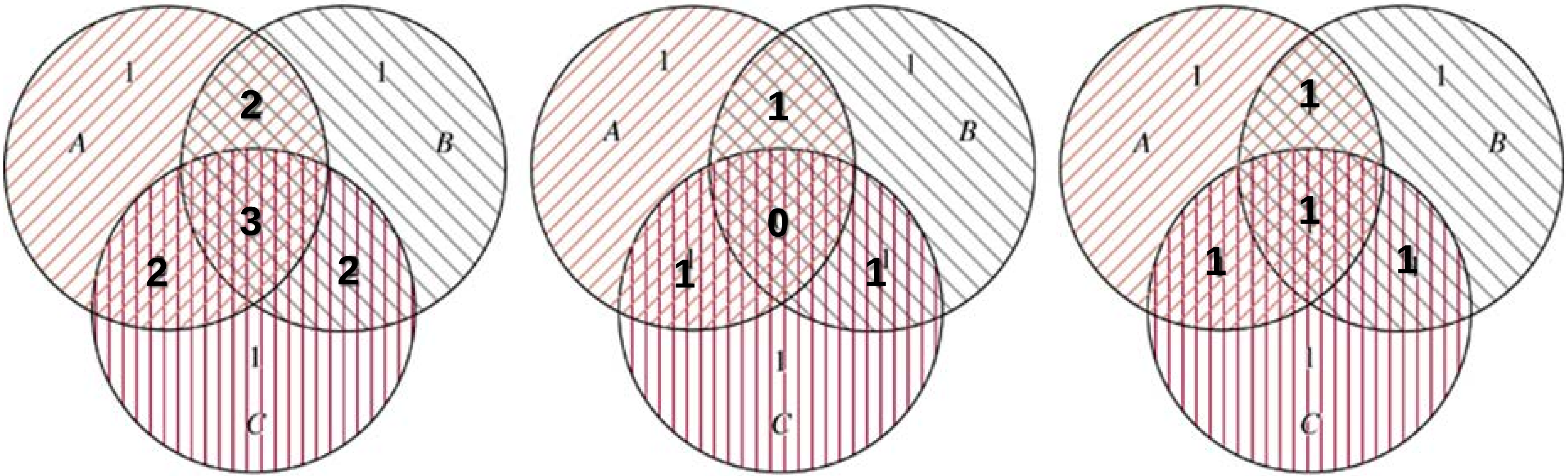


$|A| = 142$ Divisible by 7

$|A \cap B| = 12$

$|B| = 90$ Divisible by 11

Inclusion-Exclusion: Counting for 3 Sets



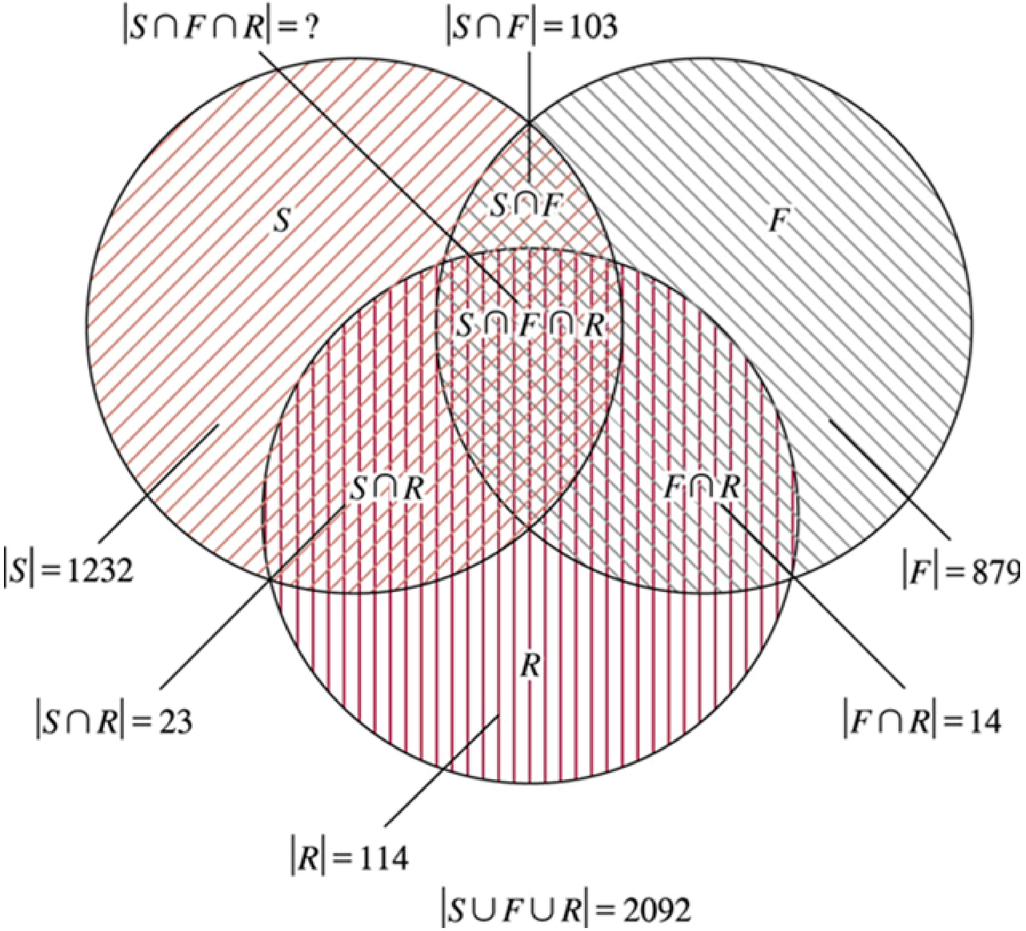
(a) Count of elements by $|A|+|B|+|C|$

(b) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|$

(c) Count of elements by $|A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$

Example

- **Q: How many students have taken courses in Spanish, French, and Russian?**
  - 1232 students take Spanish
  - 879 take French
  - 114 take Russian
  - 103 take both Spanish & French
  - 23 take both Spanish & Russian
  - 14 take both French & Russian
  - Total 2092 students



The Principle of Inclusion-Exclusion (1/2)

- **Let A_0, A_1, …, A_n be finite sets. Then**

$$|A_0 \cup A_1 \cup \cdots \cup A_n| =$$

$$\sum_{0 \le i \le n} |A_i| - \sum_{0 \le i < j \le n} |A_i \cap A_j| +$$

$$\sum_{0 \le i < j < k \le n} |A_i \cap A_j \cap A_k| + \cdots$$

$$+ (-1)^{n+1} |A_0 \cap A_1 \cap \cdots \cap A_n|$$

The Principle of Inclusion-Exclusion (2/2)

- **Pf:**

We will count the number of times for an element a such that a belongs to $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$. The element is counted C_1^r times by $\sum_i |A_i|$. It is counted C_2^r times by $\sum_{i,j} |A_i \cap A_j|$, and so on. Hence a is counted

$$C_1^r - C_2^r + C_3^r - \cdots + (-1)^{r+1} C_r^r$$

times by the right hand side. Now

$$C_0^r - C_1^r + C_2^r - C_3^r + \cdots + (-1)^r C_r^r = 0.(\text{why?})$$

We have

$$C_0^r = 1 = C_1^r - C_2^r + C_3^r - \cdots + (-1)^{r+1} C_r^r.$$

Since each element belongs to r of $A_0, A_1, \ldots, A_n$, we compute each individually and obtain the result. □

Example (1/2)

- **Find the number of primes no more than 100.**
- **Sol:**
  - Observe that for $n \le 100$, $n \ne 2, 3, 5, 7$ and n is composite if and only if $2|n$, $3|n$, $5|n$ or $7|n$. (why?)
  - Let $M_i = \{n : i|n\}$. Then the number of primes is equal to $4 + (99 – (M_2 \cup M_3 \cup M_5 \cup M_7)$.

$$|M_2| = \lfloor \frac{100}{2} \rfloor = 50 \qquad |M_2 \cap M_5| = \lfloor \frac{100}{10} \rfloor = 10$$

$$|M_3| = \lfloor \frac{100}{3} \rfloor = 33 \qquad |M_2 \cap M_7| = \lfloor \frac{100}{14} \rfloor = 7$$

$$|M_5| = \lfloor \frac{100}{5} \rfloor = 20 \qquad |M_3 \cap M_5| = \lfloor \frac{100}{15} \rfloor = 6$$

$$|M_7| = \lfloor \frac{100}{7} \rfloor = 14 \qquad |M_3 \cap M_7| = \lfloor \frac{100}{21} \rfloor = 4$$

$$|M_2 \cap M_3| = \lfloor \frac{100}{6} \rfloor = 16 \qquad |M_5 \cap M_7| = \lfloor \frac{100}{35} \rfloor = 2$$

Example (2/2)

$$|M_2 \cap M_3 \cap M_5| = \lfloor \frac{100}{30} \rfloor = 3$$

$$|M_2 \cap M_3 \cap M_7| = \lfloor \frac{100}{42} \rfloor = 2$$

$$|M_2 \cap M_5 \cap M_7| = \lfloor \frac{100}{70} \rfloor = 1$$

$$|M_3 \cap M_5 \cap M_7| = \lfloor \frac{100}{105} \rfloor = 0$$

$$|M_2 \cap M_3 \cap M_5 \cap M_7| = \lfloor \frac{100}{210} \rfloor = 0$$

- $M_2 \cup M_3 \cup M_5 \cup M_7$ = (50 + 33 + 20 + 14) - (16 + 10 + 7 + 6 + 4 + 2) + (3 + 2 + 1 + 0) - 0 = 117 - 45 + 6 = 78
- And the number of primes no more than 100 is (4 + (99 - 78)) = 25

Sieve of Eratosthenes

- **This method can be used in the previous example to compute all primes up to *n*.**
  - Define a Boolean array of size *n*.
  - Starting from 2, mark out all multiples of 2.
  - Then unmark 3 (a prime) and mark out all multiples of 3.
  - Unmark 5 is a prime, …, and so on.
- **Can you implement it?**

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Prime numbers

http://en.wikipedia.org/wiki/File:New_Animation_Sieve_of_Eratosthenes.gif

Recap: Catalan Numbers

- **Find the number of ways to parenthesize the product of n + 1 numbers x_0, x_1, …, x_n.**
- **Sol:**
  - Let C_n denote the # of ways to parenthesize the product of n+1 numbers.
  - Clearly, $C_0 = 1$.
  - $C_n = C_0C_{n-1} + C_1C_{n-2} + \ldots + C_{n-1}C_0$
  - The sequence $\{C_n\}$, $C_n = \sum_{k=0..n-1} C_kC_{n-k-1}$, is called Catalan numbers.
    - $C_1 = C_0C_0$
    - $C_2 = C_0C_1 + C_1C_0$
    - $C_3 = C_0C_2 + C_1C_1 + C_2C_0$
    - $C_4 = C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0$
    - …
    - $C_n = C_0C_{n-1} + C_1C_{n-2} + \ldots + C_{n-1}C_0$

Extended Binomial Theorem

- **Let x be a real number with $|x|<1$ and let u be a real number. Then**

$$(1+x)^u = \sum_{k=0..\infty} \binom{u}{k} x^k,$$

where $\binom{u}{k}$ is the extended binomial coefficient, u is a real number, and k is a nonnegative integer,

$$\binom{u}{k} = \begin{cases} u(u-1)\ldots(u-k+1)/k! & \text{if } k > 0 \\ 1 & \text{if } k = 0 \end{cases}$$

c.f. $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

Generating Functions

- **The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series**

$$G(x) = a_0 + a_1x + \ldots + a_kx^k + \ldots = \Sigma_{k=0..\infty}\, a_kx^k$$

Solving Catalan Numbers (1/2)

- Define a function $G(x)$ that contains all of the Catalan numbers:
  $G(x) = C_0 + C_1x + \ldots = \Sigma_{k=0..\infty} C_kx^k$
- Multiply $G(x)$ by itself to obtain $G(x)^2$
  $G(x)^2 = C_0C_0 + (C_1C_0 + C_0C_1)x + (C_2C_0 + C_1C_1 + C_0C_2)x^2 + \ldots$
- The coefficients for the powers of x are the same as those for the Catalan numbers:
  $G(x)^2 = C_1 + C_2x + C_3x^2 + \ldots$
- Multiply it by x and add C_0, we obtain
  $xG(x)^2 + C_0 = G(x) \Rightarrow xG(x)^2 - G(x) + C_0 = xG(x)^2 - G(x) + 1 = 0$.
- Hence, we have $G(x) = (1 – (1 – 4x)^{1/2})/(2x)$
  - We know that $G(0) = C_0 = 1$; if we take "+", as $x \to 0$, $G(x) \to \infty$
  - $G(x) = \dfrac{1-\sqrt{1-4x}}{2x} = \dfrac{2}{1+\sqrt{1-4x}}$

$C_1 = C_0C_0$
$C_2 = C_0C_1 + C_1C_0$
$C_3 = C_0C_2 + C_1C_1 + C_2C_0$
$C_4 = C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0$
…
$C_n = C_0C_{n-1} + C_1C_{n-2} + \ldots + C_{n-1}C_0$

Solving Catalan Numbers (2/2)

$$G(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{2}{1+\sqrt{1-4x}}$$

- $G(x)$ has a power series at 0 and its coefficients must therefore be the Catalan numbers.
- To expand $G(x)$, we use the binomial formula on $(1 – 4x)^{1/2}$

$$\sqrt{1+y} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} y^n = 1 - 2\sum_{n=1}^{\infty} \binom{2n-2}{n-1} \left(\frac{-1}{4}\right)^n \frac{y^n}{n}.$$

- Set $y = -4x$ and substitute this power series into the expression for $G(x)$ and shift the summation index n by 1, we simplify the expansion to

$$G(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1}.$$

- Hence, $C_n = \frac{1}{n+1}\binom{2n}{n}$