國立交通大學電子工程學系

CHAPTER 8

RELATIONS

Outline

- **Content**
  - Relations and their properties
  - *n*-ary Relations and their applications
  - Representing relations
  - Closures of relations
  - Equivalence relations
  - Partial orderings
- **Reading**
  - Chapter 8

Most of the following slides are by courtesy of Prof. H.-C. Yen and Prof. J.-D. Huang

Relations and Their Properties

- **Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.**
  - Let R be a binary relation. We sometimes write aRb for $(a, b) \in R$.
  - Examples of binary relations are $<$, $\in$, etc.
  - Functions belong to a special class of relations. Let $f: A \rightarrow B$. Define $F = \{(a, b) : b = f(a)\}$. F is a binary relation from A to B such that aFb and aFb' implies $b = b'$.

- **Let A be a set. A relation on A is a relation from A to A.**
  - For instance, $<$, $=$ are relation on **R**.

Relations and Their Properties

- **A relation R on A is called reflexive if $(a, a) \in R$ for all $a \in A$.**
  - For instance, $\leq$ is reflexive.

- **A relation R on A is called symmetric if $(a, b) \in R$ implies $(b, a) \in R$ for all $a, b \in A$.**
- **A relation R on A is called antisymmetric if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$.**
  - For instance, $\neq$ is symmetric and $\geq$ is antisymmetric.
  - Antisymmetric $\neq$ not symmetric

- **A relation R on A is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.**
  - For instance, $<$ is transitive.

Relations and Their Properties

- **Let R be a relation from A to B and S is a relation from B to C. The composite of R and S, $S \circ R$, is defined by**

  $$S \circ R = \{(a, c) : \exists b \in B, (a, b) \in R \wedge (b, c) \in S\}.$$

  - When R and S happen to be functions, $S \circ R$ is equivalent to the function composition.

- **Let R be a relation on A. Define**

  $$R^1 = R \text{ and } R^{n+1} = R^n \circ R.$$

  For convenience, we define $R^0 = I$ where I is the identity relation $\{(a, a) : a \in A\}$.

Relations and Their Properties

- **Theorem: The relation R on A is transitive if and only if $R^n \subseteq R$ for $n \in Z^+$.**
- **Pf:**

(⇐) Let $a, b, c \in A$ with aRb and bRc. Then aR^2c by definition. Hence aRc for $R^2 \subseteq R$.
(⇒) We prove by induction on n.
BASIS STEP: $n = 1$, $R \subseteq R$ is trivial.
INDUCTIVE STEP: Assume $R^k \subseteq R$. We want to show $R^{k+1} \subseteq R$. Consider any a, c such that $aR^{k+1}c$. There is a b such that aRb and bR^kc because $R^{k+1} = R^k \circ R$. By inductive hypothesis, bR^kc implies bRc. Hence aRc follows from the transitivity of R. □

n-ary Relations

- **Relations among elements of more than 2 sets**
  - These relationships are called *n*-ary relations
- **Let $A_0, A_1, \ldots, A_{n-1}$ be sets. An *n*-ary relation on these sets is a subset of $A_0 \times A_1 \times \ldots \times A_{n-1}$. $A_0, A_1, \ldots, A_{n-1}$. The sets $A_0, A_1, \ldots, A_{n-1}$ are called the domain of the relation and n is its degree.**
- **E.g., Let R be the relation on N×N×N×N consisting of 4-tuple (a, b, c, d), where $a, b, c,$ and d are integers with $a < b < c < d$. Then $(2, 3, 5, 6) \in R$, but $(4, 1, 8, 3) \notin R$. The degree of the relation is 4 and its domains are all equal to N.**
- **E.g., Let R be the relation on $Z \times Z \times Z^+$ consisting of triples (a, b, m), where $a, b,$ and m are integers with $m > 0$ and $a \equiv b \pmod{m}$. Then $(8, 2, 3) \in R$, but $(-1, -8, 5) \notin R$. The degree of the relation is 3. Its first two domains are equal to Z and its third domain is the set of positive integers (Z^+).**

Databases and Relations

- **The relational data model represents a database of records as an *n*-ary relation**
- **E.g., a student database of six records which are represented as 4-tuples of the form (student name, id number, Major, GPA) are**

(Ackermann, 231455, Computer Science, 3.88) (Adams, 888323, Physics, 3.45)
(Chou, 102147, Computer Science, 3.49) (Goodfriend, 453876, Mathematics, 3.45)
(Rao, 678543, Mathematics, 3.90) (Stevens, 786576, Psychology, 2.99)

- **Relations used to represent databases are also called tables, because these relations are often displayed as tables.**
- **E.g., student database**

TABLE 1 Students.

Student_name	*ID_number*	*Major*	*GPA*
Ackermann	231455	Computer Science	3.88
Adams	888323	Physics	3.45
Chou	102147	Computer Science	3.49
Goodfriend	453876	Mathematics	3.45
Rao	678543	Mathematics	3.90
Stevens	786576	Psychology	2.99

Selection Operator

- **Let R be an n-ary relation over $A_0, A_1, \ldots, A_{n-1}$ and C be a condition that elements in R may satisfy.**
  The selection operator s_C: maps R to the n-ary relation of all n-tuples from R that satisfy the condition C.
  $$s_C(R) = \{(a_0, a_1, \ldots, a_{n-1}) \in R : C(a_0, a_1, \ldots, a_{n-1}) = \text{true}\}.$$

- **E.g., To find the records of Mathematics majors in the n-ary relation R shown in Table1, we can use the operator s_{C1}, where C1 is the condition (Major = Mathematic).**
  - The result is the two 4-tuples (Goodfriend, 453876, Mathematics, 3.45) and (Rao, 678543, Mathematics, 3.90).

- **Similarly, to find the records of Mathematics majors who have a GPA above 3.5, we can use the operator s_{C2}, where C2 is the condition (Major = Mathematic $\wedge$ GPA > 3.5).**
  - The result is the 4-tuples (Rao, 678543, Mathematics, 3.90).

Projection operator

- **The projection $P_{i0,i1, \ldots ,im-1}$ maps the n-tuple $(a_0, a_1, \ldots, a_{n-1})$ to the m-tuple $(a_{i0}, a_{i1}, \ldots, a_{im-1})$, where $m \leq n$ and $0 \leq i_k < n$ for all k.**

- **E.g., what results when the projection $P_{0,2}$ is applied to the 4-tuples (2, 3, 0, 4) and (Adams, 888323, Physics, 3.45)?**
  - The projection $P_{1,3}$ sends these 4-tuples to (2, 0) and (Adams, Physics) respectively.

Join operator

- **Let R be a relation of degree m and S a relation of degree n. The join $J_p(R, S)$, where $p \leq m$ and $p \leq n$, is a relation of degree $m + n - p$ such that**

$$(a_0, a_1, \ldots, a_{m-p-1}, c_0, c_1, \ldots, c_{p-1}, b_p, b_{p+1}, \ldots, b_{n-1}) \in J_p(R, S)$$

if and only if

$$(a_0, a_1, \ldots, a_{m-p-1}, c_0, c_1, \ldots, c_{p-1}) \in R$$

and

$$(c_0, c_1, \ldots, c_{p-1}, b_p, b_{p+1}, \ldots, b_{n-1}) \in S.$$

Join operator

- **E.g. what relation results when the join operator J_2 is used to combine the relation displayed in the Table5 and 6.**

TABLE 5 Teaching_assignments.

Professor	*Department*	*Course_ number*
Cruz	Zoology	335
Cruz	Zoology	412
Farber	Psychology	501
Farber	Psychology	617
Grammer	Physics	544
Grammer	Physics	551
Rosen	Computer Science	518
Rosen	Mathematics	575

TABLE 6 Class_schedule.

Department	*Course_ number*	*Room*	*Time*
Computer Science	518	N521	2:00 P.M.
Mathematics	575	N502	3:00 P.M.
Mathematics	611	N521	4:00 P.M.
Physics	544	B505	4:00 P.M.
Psychology	501	A100	3:00 P.M.
Psychology	617	A110	11:00 A.M.
Zoology	335	A100	9:00 A.M.
Zoology	412	A100	8:00 A.M.

J_2 ↓

TABLE 7 Teaching_schedule.

Professor	*Department*	*Course_number*	*Room*	*Time*
Cruz	Zoology	335	A100	9:00 A.M.
Cruz	Zoology	412	A100	8:00 A.M.
Farber	Psychology	501	A100	3:00 P.M.
Farber	Psychology	617	A110	11:00 A.M.
Grammer	Physics	544	B505	4:00 P.M.
Rosen	Computer Science	518	N521	2:00 P.M.
Rosen	Mathematics	575	N502	3:00 P.M.

Representing Relations - Matrix

- **We can represent relations by matrices or graphs.**

- **Consider any binary relation R on $\{a_0, a_1, \ldots, a_{n-1}\}$. Define $\mathbf{M}_R = [m_{ij}]_{nxn}$ where**

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, a_j) \in R \\ 0 \text{ if } (a_i, a_j) \notin R \end{cases}$$

  - If R is symmetric, $\mathbf{M}_R$ is symmetric. If R is reflexive and $\mathbf{M}_R = [m_{ij}]_{nxn}$, $m_{ii} = 1$ for $0 \leq i < n$.

Matrix Representation

- **We introduce the following matrix operations:**

- **Let A = $[a_{ij}]_{mxn}$, B = $[b_{ij}]_{mxn}$ and a_{ij}, $b_{ij} \in \{0, 1\}$ for $0 \leq i < m$, $0 \leq j < n$. Define A ∧ B = $[c_{ij}]_{mxn}$ where**

$$c_{ij} = \begin{cases} 1 \text{ if } a_{ij} = 1 \wedge b_{ij} = 1 \\ 0 \text{ otherwise} \end{cases}$$

- **Let A = $[a_{ij}]_{mxn}$, B = $[b_{ij}]_{mxn}$ and a_{ij}, $b_{ij} \in \{0, 1\}$ for $0 \leq i < m$, $0 \leq j < n$. Define A ∨ B = $[c_{ij}]_{mxn}$ where**

$$c_{ij} = \begin{cases} 1 \text{ if } a_{ij} = 1 \vee b_{ij} = 1 \\ 0 \text{ otherwise} \end{cases}$$

Matrix Representation

- **Let A = $[a_{ik}]_{mxl}$, B = $[b_{kj}]_{lxn}$ and a_{ik}, $b_{kj} \in \{0, 1\}$ for $0 \le i < m$, $0 \le k < l$, $0 \le j < n$. Define A ⊙ B = $[c_{ij}]_{mxn}$ where**

$$c_{ij} = \begin{cases} 1 \text{ if } \bigvee_{k=0..l-1} a_{ik} = 1 \wedge b_{kj} = 1 \\ 0 \text{ otherwise} \end{cases}$$

When $m = l$, define $A^{[0]} = I_n$ and $A^{[r+1]} = A \odot A^{[r]}$

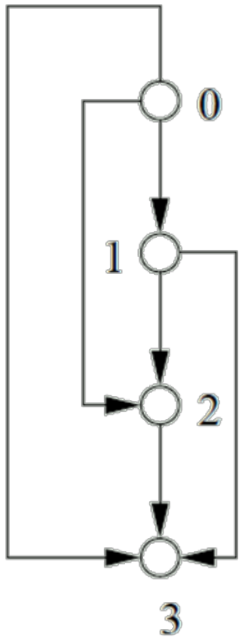
- $M_{SoR} = M_R \odot M_S$
  - $SoR = \{(a, c) : \exists b \in B, (a, b) \in R \wedge (b, c) \in S\}$.
  - $c_{ij} = 1$ iff $a_{ik} = b_{kj} = 1$ for some k; $M_{SoR} = M_R \odot M_S$
- Let R_0 and R_1 be relations on $\{0, \ldots, n\text{-}1\}$. It is straightforward to see

  $M_{R0\cup R1} = M_{R0} \vee M_{R1}$, $M_{R0\cap R1} = M_{R0} \wedge M_{R1}$ and $M_{R1oR0} = M_{R0} \odot M_{R1}$.
  Note that the order of R_0 and R_1 is reversed in the composite.

Graph Representation

- **A directed graph (or digraph), $G = (V, E)$, consists of the set V of vertices and $E \subseteq V \times V$ the set of edges. For the edge (a, b), the vertex a is its initial vertex and b its terminal vertex. The edge (a, a), is called a loop.**

- **E.g., draw a digraph to represent < on {0, 1, 2, 3}.**



Closures of Relations

- **Let *R* be a relation on *A*. The smallest transitive relation that contains *R* is called the transitive closure of *R*.**
- **Similarly, the smallest reflexive relation that contains *R* is called the reflexive closure of *R*.**
- **And the smallest symmetric relation that contains *R* is called the symmetric closure of *R*.**

- **The relation $\Delta_A = \{(a, a) : a \in A\}$ is the diagonal relation on *A*. Note that when $|A| = n$, $\mathbf{M}_{\Delta_A} = \mathbf{I}_n$**

Example: Reflexive Closure

- **R = {(1,1), (1,2), (2,1), (3,2)} on the set A = {1,2,3}**
  - R is not reflexive
- **Let $S = R \cup \{ (2,2), (3,3) \}$**
  - S is reflexive
- **S is the reflexive closure of R**

- **Let $\Delta = \{(a, a) \mid a \in A \}$,**
- **The reflexive closure of R equals $R \cup \Delta$**

Example: Symmetric Closure

- **Let R^{-1} = {(b, a) | (a, b) ∈ R }**
  - the inverse of R
- **The symmetric closure of R equals $R \cup R^{-1}$**

- **What is the symmetric closure of the relation R = {(a, b) | $a < b$, a, b ∈ Z } ?**
  - R^{-1} should be {(a, b) | $a > b$, a, b ∈ **Z**}
  - $R \cup R^{-1}$ = {(a, b) | $a \neq b$, a, b ∈ **Z**}

Example: Transitive Closure

- **R = {(1,3), (1,4), (2,1), (3,2)} on {1,2,3,4}**
  - R is not transitive
- **Add { (1,2), (2,3), (2,4), (3,1) } ➔**
- **S = {(1,3), (1,4), (2,1), (3,2), (1,2), (2,3), (2,4), (3,1) }**
  - *S* is still **not transitive** !!!

- **How to get the transitive closure of R?**

Closures of Relations

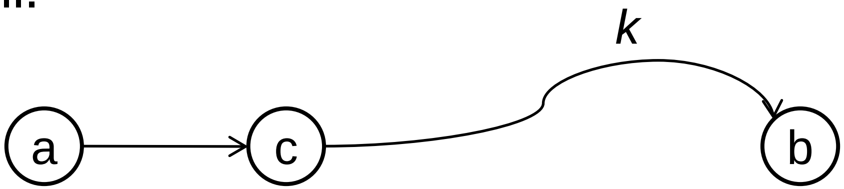
- **Let $G = (V, E)$ be a digraph. A path from v_0 to v_n is a sequence of edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ in G. We usually write $v_0, v_1, \ldots, v_n$ to denote the path and say the length of path is n.**

- **For any $v \in V$, we view the empty sequence as a path (of length 0). If $n > 0$ and $v_0 = v_n$, we say the path is a circuit or cycle.**

- **Just like a relation can be represented by a digraph. A digraph corresponds to a relation E on V.**

Closures of Relations

- **Theorem: Let *R* be a relation on a set *A*. Consider the digraph (*A, R*). There is a path of length $n > 0$ from *a* to *b* iff $(a, b) \in R^n$**
- **Pf:**
  - We prove by induction.
  - Basis step: $n = 1$. Obvious.
  - Inductive step: Assume there is a path of length *k* from *c* to *b* if and only if $(c, b) \in R^k$.
  - Consider any path of length $k + 1$ from *a* to *b*. The path consists a path of length 1 from *a* to *c* and a path of length *k* from *c* to *b*. Hence, $(a, c) \in R$ and $(c, b) \in R^k$. We have $(a, b) \in R^{k+1}$.
  - On the other hand, if $(a, b) \in R^{k+1}$, there exists a *c* such that $(a, c) \in R$ and $(c, b) \in R^k$. The result follows from the inductive hypothesis as well.



Connectivity Relation

- **Let R be a relation on A. The connectivity relation R^+ consists of (a, b) such that there is a path from a to b in the digraph (A, R).**
  - $R^+ = \bigcup_{n=1}^{\infty} R^n$

- **Theorem: The transitive closure of R equals to R^+**
- **Pf:**
  - Note that $R \subseteq R^+$ by definition. Furthermore, (a, b), (b, c) $\in R^+$ implies (a, b) $\in R^i$ and (b, c) $\in R^j$ for some i, j.
  - Hence (a, c) $\in R^{i+j} \subseteq R^+$. R^+ is transitive.
  - It remains to show that any transitive relation containing R must contain R^+.
  - Let S be any transitive relation containing R. Since S is transitive, $S^n \subseteq S$ for $n \in Z^+$.
  - Therefore, $R^+ = \bigcup_{n=1}^{\infty} R^n \subseteq \bigcup_{n=1}^{\infty} S^n \subseteq S$.

Connectivity Relations

- **Lemma: Let *A* be a set with |*A*| = *n* and *R* a relation on *A*. If there is a path of length > 0 from *a* to *b* in the digraph (*A*, *R*), then there is a path of length ≤ *n*. Moreover, when *a* ≠ *b*, if there is a path from *a* to *b* of length > 0 in (*A*, *R*), then there is such a path of length < *n*.**
- **Pf:**

Consider the shortest path $v_0 = a, v_1, v_2, \ldots, v_m = b$ from a to b of length m. If $v_0 = v_m$ and $m > n$, there must be some i, j with $i < j$ such that $v_i = v_j$ by the pigeonhole principle. Then $v_0, v_1, \ldots, v_i, v_{j+1}, v_{j+2}, \ldots, v_m$ is a shorter path from a to b. A contradiction.

Now suppose $a = v_0 \neq v_m = b$ and $m \geq n$. There must be i, j with $i < j$ such that $v_i = v_j$. We can also construct a shorter path and lead to a contradiction. □

Closures of Relations

- **Lemma: Let $M_R = [m_{ij}]_{nxn}$ be the matrix of the relation *R* on a set with *n* elements. Then the matrix M_{R+} of R^+ is**

$$\mathbf{M}_{R^+} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n]}.$$

- **Pf:**
  - $(a, b) \in R^+$, then there is a path from a to b by definition.
  - By the above Lemma, it suffices to consider paths of length at most *n*. Recall that

$$R^+ = R \cup R^2 \cup \cdots \cup R^n.$$

  - The result follows from the matrix representation of relations.

Computing Transitive Closure

procedure transitive-closure(M$_R$: zero-one *n* x *n* matrix)

1. **A** := **M**$_R$
2. **B** := **A**
3. **for** *i* := 2 **to** *n* **do**
4. **A** := **A** ⊙ **M**$_R$
5. **B** := **B** ∨ **A**
6. { **B** is the zero-one matrix for R^+}

$O(n^4)$

Warshall's Algorithm

procedure Warshall (M_R: zero-one n x n matrix)

W := $\mathbf{M}_R$

for k := 0 **to** n -1 **do**

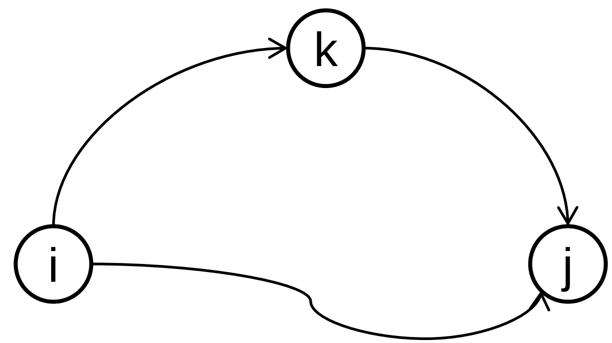
  for i := 0 **to** n -1 **do**

    for j := 0 **to** n -1 **do**

      $w_{ij} := w_{ij} \vee (w_{ik} \wedge w_{kj})$

{**W** = $[w_{ij}]$ is $\mathbf{M}_R$}

O(n^3)



$$w_{ij}^{[k]} = w_{ij}^{[k-1]} \vee (w_{ik}^{[k-1]} \wedge w_{kj}^{[k-1]}).$$

We can rephrase it as the equivalence of the following two statements:

- There is a path from v_i to v_j via $\{v_0, v_1, \ldots, v_k\}$;
- There is a path from v_i to v_j via $\{v_0, v_1, \ldots, v_{k-1}\}$ or a path from v_i to v_k and v_k to v_j via $\{v_0, v_1, \ldots, v_{k-1}\}$.

Equivalence Relations

- **A relation *R* on *A* is called an equivalence relation if it is reflexive, symmetric, and transitive.**

- **Two elements that are related by an equivalence relation are called equivalent**
  - *a* is equivalent to *a*
  - If *a* and *b* are equivalent, and *b* and *c* are equivalent, then *a* and *c* are equivalent

Example

Equivalence: reflexive, symmetric, and transitive

- **Suppose that *R* is the relation on the set of strings of English letters such that *aRb* iff *l*(*a*) = *l*(*b*). Is *R* an equivalence relation?**

- **Let *R* be the relation on the set of integers such that *aRb* iff *a* = *b* or *a* = - *b*. Is *R* an equivalence relation?**

- **Let *R* be the relation of real numbers such that *aRb* iff *a*-*b* is an integer. Is *R* an equivalence relation?**

Example

Equivalence: reflexive, symmetric, and transitive

- **Let $n \in Z^+$. Show $R = \{(a, b) : a = b \pmod{n}\}$ is an equivalence relation on the set of integers.**

- **Pf:**
  - For any a, b, $c \in$ **N**, we have
  - $a = a \pmod{n}$;
  - If $a = b \pmod{n}$; Then $n|a\text{-}b$. Thus $n|b\text{-}a$. So $b = a \pmod{n}$;
  - If $a = b \pmod{n}$ and $b = c \pmod{n}$, then $nk = a - b$ and $nk' = b - c$. So $n(k + k') = (a - b) + (b - c) = a\text{-}c$, $n|a\text{-}c$. We have $a = c \pmod{n}$.

Equivalence Classes

- **Let R be an equivalence relation on a set A. Let $a \in A$. Define the equivalence class of a, $[a]_R$, to be $[a]_R = \{b \mid (a, b) \in R\}$**
  - Sometimes, we may write $[a]$ if R is clear from context.

- **If $b \in [a]_R$, then b is called a representative of this equivalence class**
  - **Any** element in this class can be a representative

Example

- **Let *R* be the relation on the set of integers such that *aRb* iff *a* = *b* or *a* = - *b*. What is the equivalence class of a given integer?**

- **Sol:**
  - [7] = {7, -7}
  - [5] = {5, -5}
  - [0] = {0}

Example

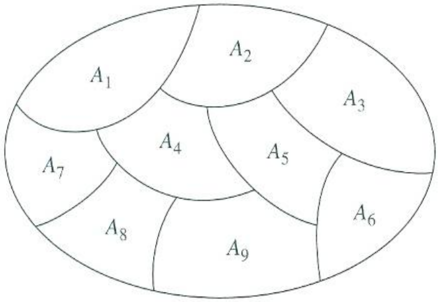
- **Find all equivalence classes for R = {(a, b) : a = b (mod n)}**
- **Sol:**

$$
\begin{aligned}
[0] &= \{\ldots, -2n, -n, 0, n, 2n, \ldots\} \\
[1] &= \{\ldots, -2n+1, -n+1, 1, n+1, 2n+1, \ldots\} \\
&\ldots \\
[n-1] &= \{\ldots, -2n+(n-1), -n+(n-1), (n-1), \\
&\quad n+(n-1), 2n+(n-1), \ldots\}
\end{aligned}
$$

- **What are the equivalence class of 0 and 1 for congruence modulo 4?**
  - [0] = { …, -8, -4, 0, 4, 8, …}
  - [1] = { …, -7, -3, 1, 5, 9, …}
- **The equivalence classes of the relation congruence modulo n are called congruence classes modulo n.**

Properties of Equivalence Classes

- Let A be a set. A partition of A is a collection of disjoint nonempty subsets of A.
  The equivalence classes of R on A form a partition of A.



- Theorem: Let R be an equivalence relation on A. The following statements are equivalence:
  1. aRb;
  2. $[a] = [b]$;
  3. $[a] \cap [b] \neq \emptyset$.

I is an index set

- Theorem: Let A be a set and $\{A_i : i \in I\}$ a partition of A. Define
  $$R = \{(a, b) : \exists\, i, a \in A_i \wedge b \in A_i\}$$
  Then R is an equivalence relation with A_i, $i \in I$ its equivalence classes.

Partial Orderings

A relation R on A is called antisymmetric if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$.

- **Let R be a relation on A. R is called partial ordering or partial order if it is reflexive, antisymmetric, and transitive.**
- **The set A is called a partially ordered set, or poset, and is denoted by (A, R).**
- **E.g.,**
  - $(\mathbf{Z}, \geq)$ is a poset.
  - $(P(S), \subseteq)$ is a poset for any set S.

Posets

- **In a poset (A, R)**
  - The notation $a \preccurlyeq b$ denotes that $(a, b) \in R$
  - $\preccurlyeq$ is used to denote the relation in any poset, not just "$\leq$" relation
  - the notation $a \prec b$ denotes that $a \preccurlyeq b$ but $a \neq b$
- **When a and b are elements of the poset (A, $\preccurlyeq$) , it is not necessary that either $a \preccurlyeq b$ or $b \preccurlyeq a$**
  - e.g., in (P(**Z**), $\subseteq$), {1, 2} is not related to {1, 3}
  - that is, neither $\{1,2\} \subseteq \{1,3\}$ nor $\{1,3\} \subseteq \{1,2\}$

Incomparable

- **The elements *a* and *b* of a poset are called comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$**
- **The elements *a* and *b* of a poset are called incomparable if neither $a \preccurlyeq b$ nor $b \preccurlyeq a$**
- **E.g., in the poset (Z^+, |), are 3 and 9 comparable? are 5 and 7 comparable?**
  - 3 and 9 are comparable while 5 and 7 are incomparable
- **Why partial ordering?**
  - because some pairs of elements may be incomparable

Total Ordering

- **If every 2 elements in the set are comparable, the relation is called a total ordering**
- **If (S, $\preccurlyeq$) is a poset and every 2 elements of S are comparable, then**
  - S is called a **totally/linearly ordered set**
  - $\preccurlyeq$ is called a **total/linear order**
- **A totally ordered set is also called a chain**

- **Let (S, $\preccurlyeq$) be a poset. (S, $\preccurlyeq$) is a well-ordered set if $\preccurlyeq$ is a total ordering such that every nonempty subset of S has a least element (according to $\preccurlyeq$).**

- **E.g.,**
  - ($\mathbf{Z}^+$, $\leq$) is well-ordered, but ($\mathbf{Z}$, $\leq$) is not

Lexicographic Ordering

- **The word in a dictionary are listed in alphabetic or lexicographic order**
- **Let (A_1, $\preccurlyeq_1$) and (A_2, $\preccurlyeq_2$) be 2 posets. Then the lexicographic ordering $\preccurlyeq$ on A_1 x A_2 is defined by specifying that one pair is less than a second pair if**
  - the 1st entry of the 1st pair is less than that of the 2nd pair,
  - or, the 1st entries are equal, but the 2nd entry of the 1st pair is less than that of the 2nd pair
- **In other words, $(a_1, a_2) \prec (b_1, b_2)$ if**
  - $a_1 \prec_1 b_1$, or
  - $a_1 = b_1$ and $a_2 \prec_2 b_2$

Hasse Diagrams

/ˈhæsə/

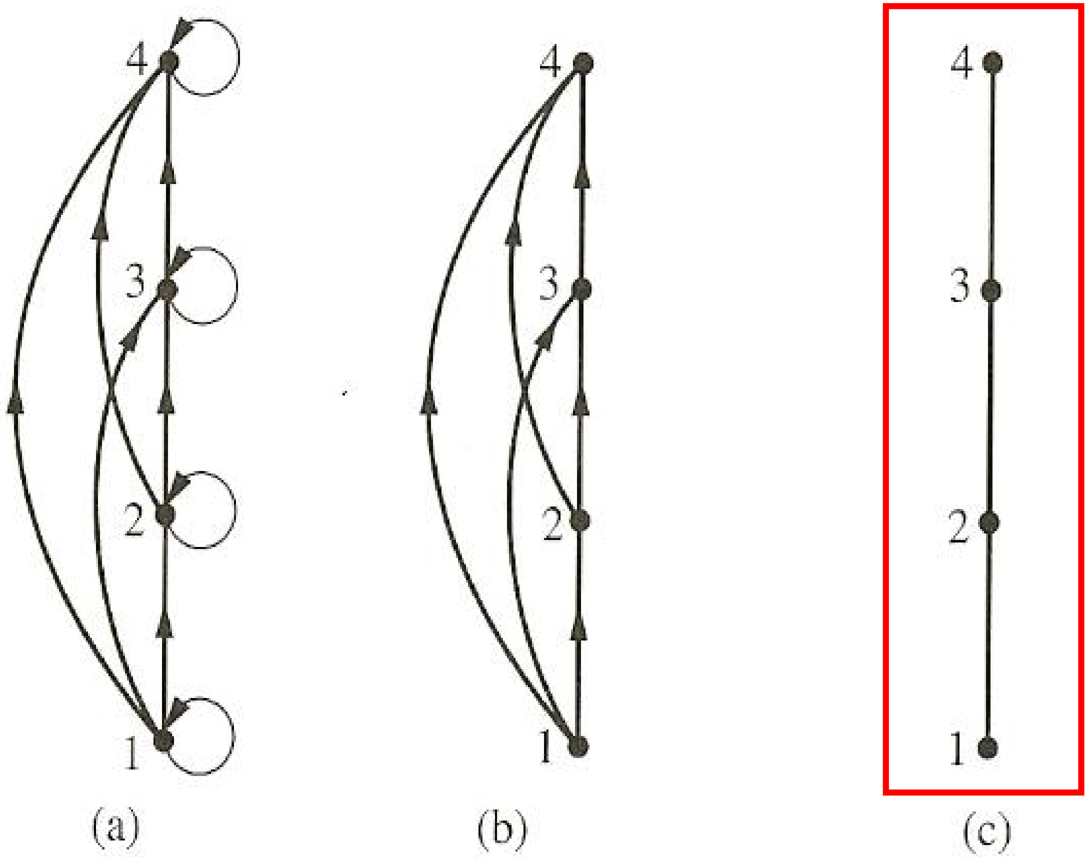


FIGURE 2 **Constructing the Hasse Diagram for ({1, 2, 3, 4}, ≤).**

Hasse Diagrams

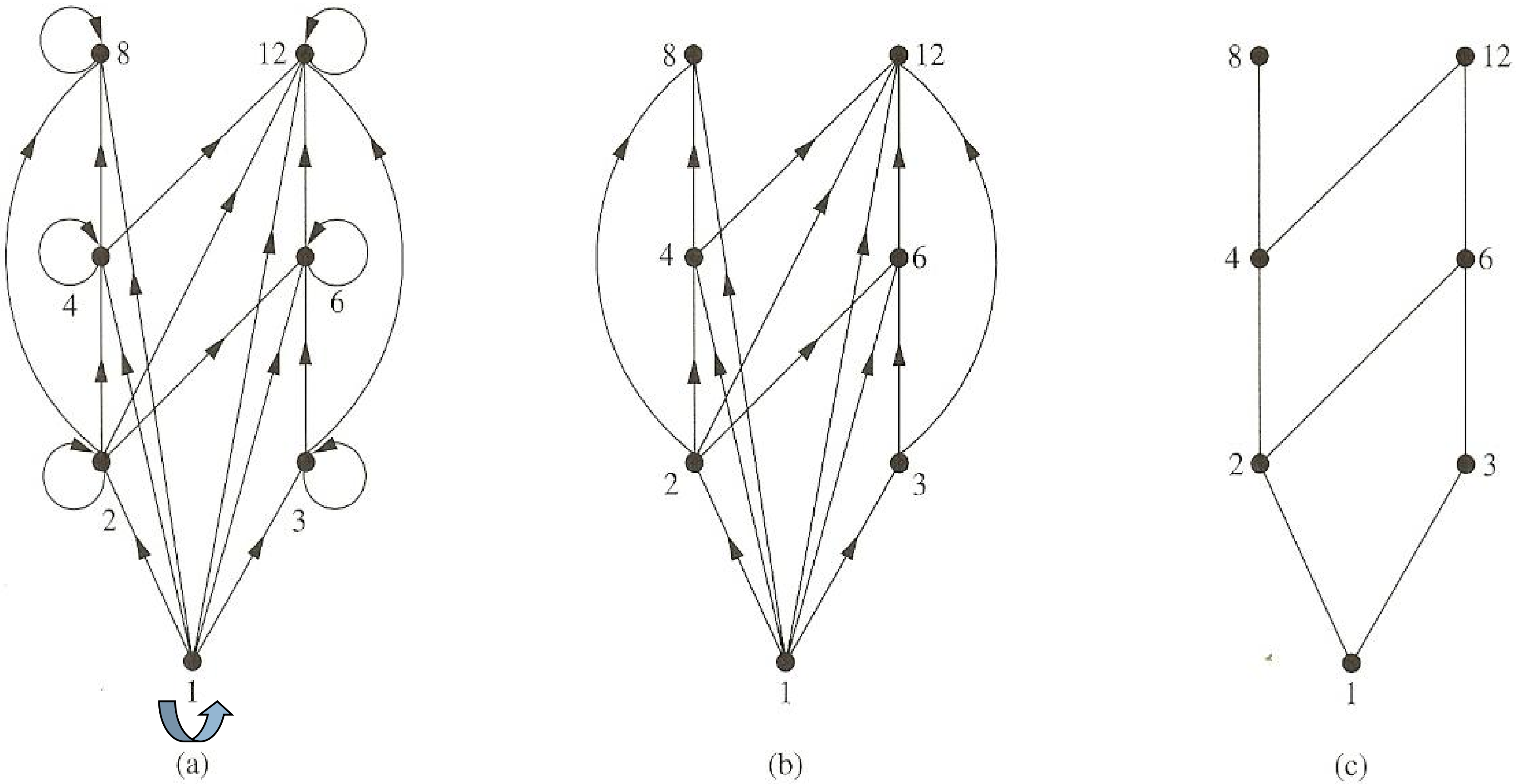


FIGURE 3 **Constructing the Hasse Diagram of ({1, 2, 3, 4, 6, 8, 12}, |).**

Maximal and Minimal Elements

- **An element of a poset is called maximal if it is not less than any element of the poset**
  - that is, a is maximal in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \prec b$
- **An element of a poset is called minimal if it is not greater than any element of the poset**
  - that is, a is minimal in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $b \prec a$
- **Maximal/minimal elements in Hasse diagrams?**

Example

- Find the maximal/minimal elements

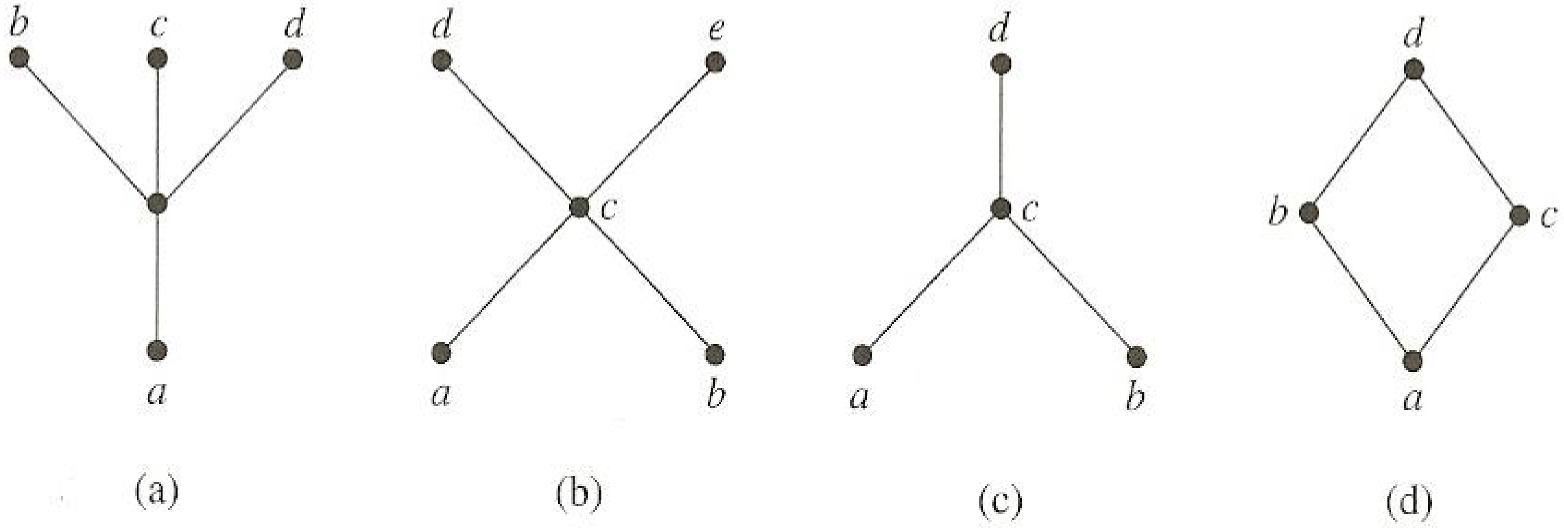


FIGURE 6 Hasse Diagrams of Four Posets.

- A poset can have more than 1 maximal/minimal elements

Greatest and Least Elements

- **There is an element in a poset that is greater than every other element**
  - greatest element
  - a is the greatest element of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$
- **There is an element in a poset that is less than every other element**
  - least element
  - a is the least element of the poset $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$
- **Greatest/least element is unique when it exists**

Example

- Find the greatest/least elements

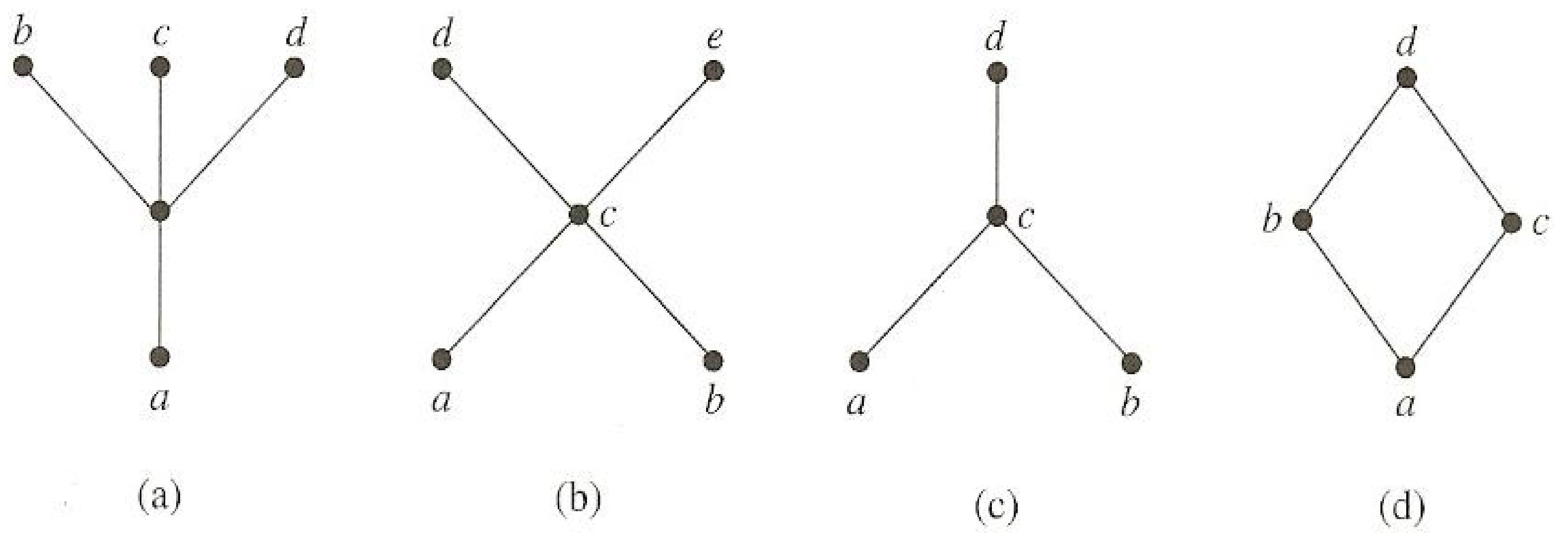


FIGURE 6 Hasse Diagrams of Four Posets.

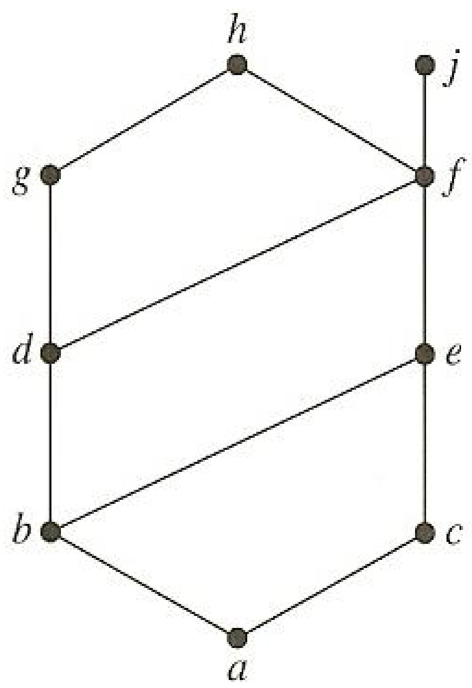
Example

- **Let *S* be a set. Determine whether there is a greatest/least element in the poset (P(*S*), $\subseteq$)**

- **Is there a greatest/least element in the poset (Z^+, |)?**

Upper Bound and Lower Bound

- **Assume**
  - $(S, \preccurlyeq)$ is a poset and $A \subseteq S$
  - $u \in S$ is called an **upper bound** of A if $\forall a \in A$, $a \preccurlyeq u$
  - $l \in S$ is called a **lower bound** of A if $\forall a \in A$, $l \preccurlyeq a$
- **E.g.,**
  - Find the lower/upper bound of the set $\{a, b, c\}$, $\{j, h\}$, $\{a, c, d, f\}$, respectively



Least Upper Bound

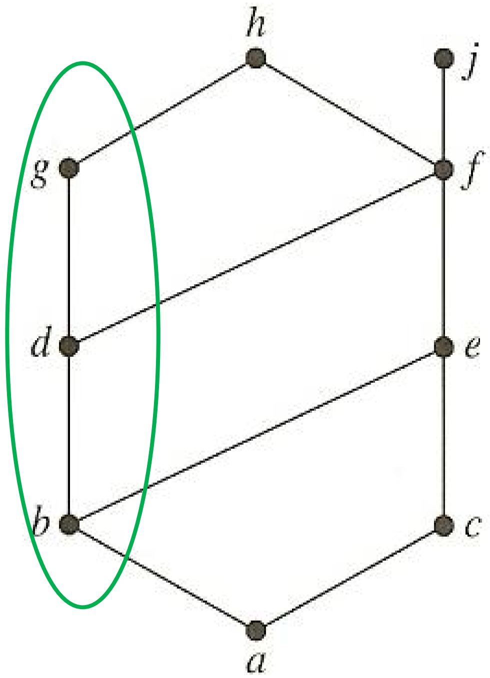
- **The element *x* is called the least upper bound of the subset *A* if *x* is an upper bound that is less than every other upper bound of *A***
  - denoted as lub(*A*)
  - that is, *x* is an upper bound of *A*, and $x \preccurlyeq z$ whenever *z* is an upper bound of *A*
  - it's unique if it exists

Greatest Lower Bound

- **The element *y* is called the greatest lower bound of the subset *A* if *y* is a lower bound that is greater than every other lower bound of *A***
  - denoted as glb(*A*)
  - that is, *y* is a lower bound of *A*, and $z \preccurlyeq y$ whenever *z* is a lower bound of *A*
  - it's unique if it exists (why?)

Examples of lub and glb

- **E.g.,**
  - Find glb and lub of {b,d,g} if they exist

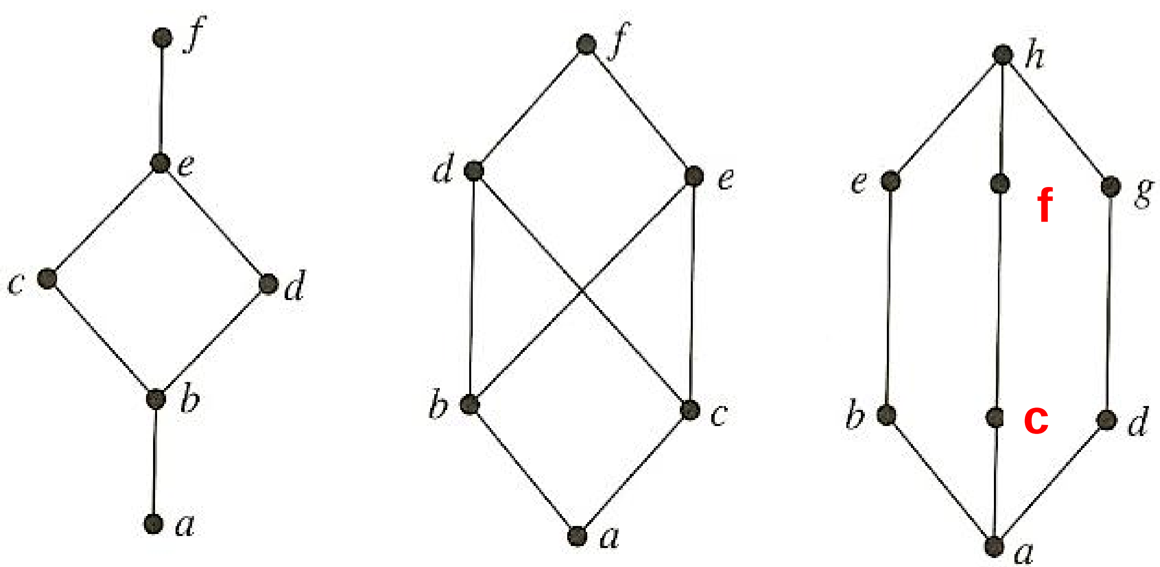


  - Find glb and lub of (P(*S*), ⊆) if they exist

∅ and *S*

Lattices

- **Lattice**
  - A poset in which **every pair** of elements has both a least upper bound and a greatest lower bound
- **E.g.,**
  - Is the poset (Z^+, |) a lattice? lub: lcm(a, b); glb: gcd(a, b)
  - Is the poset ($P(S)$, $\subseteq$) a lattice? glb: A $\cap$ B; lub: A $\cup$ B



Topological Sorting (1/2)

- **How to complete a project made up of 20 different tasks step by step?**
  - set up a partial order on the set of tasks, so that $a \prec b$ iff b cannot be started until a has been completed
  - produce an order of 20 tasks that is compatible with this partial order
  - E.g., precedence constraint, priority, data dependency
- **A total ordering $\preccurlyeq$ is compatible with the partial ordering R if $a \preccurlyeq b$ whenever aRb**
- **Constructing a compatible total ordering from a partial ordering is called topological sorting**

Topological Sorting (2/2)

- **Lemma: Every finite nonempty poset (S, $\preccurlyeq$) has a minimal element**

ALGORITHM 1 Topological Sorting

procedure *topological sort* (S: finite poset)
$k := 1$
while $S \neq \emptyset$
begin
    $a_k := $ a minimal element of S {such an element exists by Lemma 1}
    $S := S - \{a_k\}$
    $k := k + 1$
end $\{a_1, a_2, \ldots, a_n$ is a compatible total ordering of $S\}$

Examples (1/2)

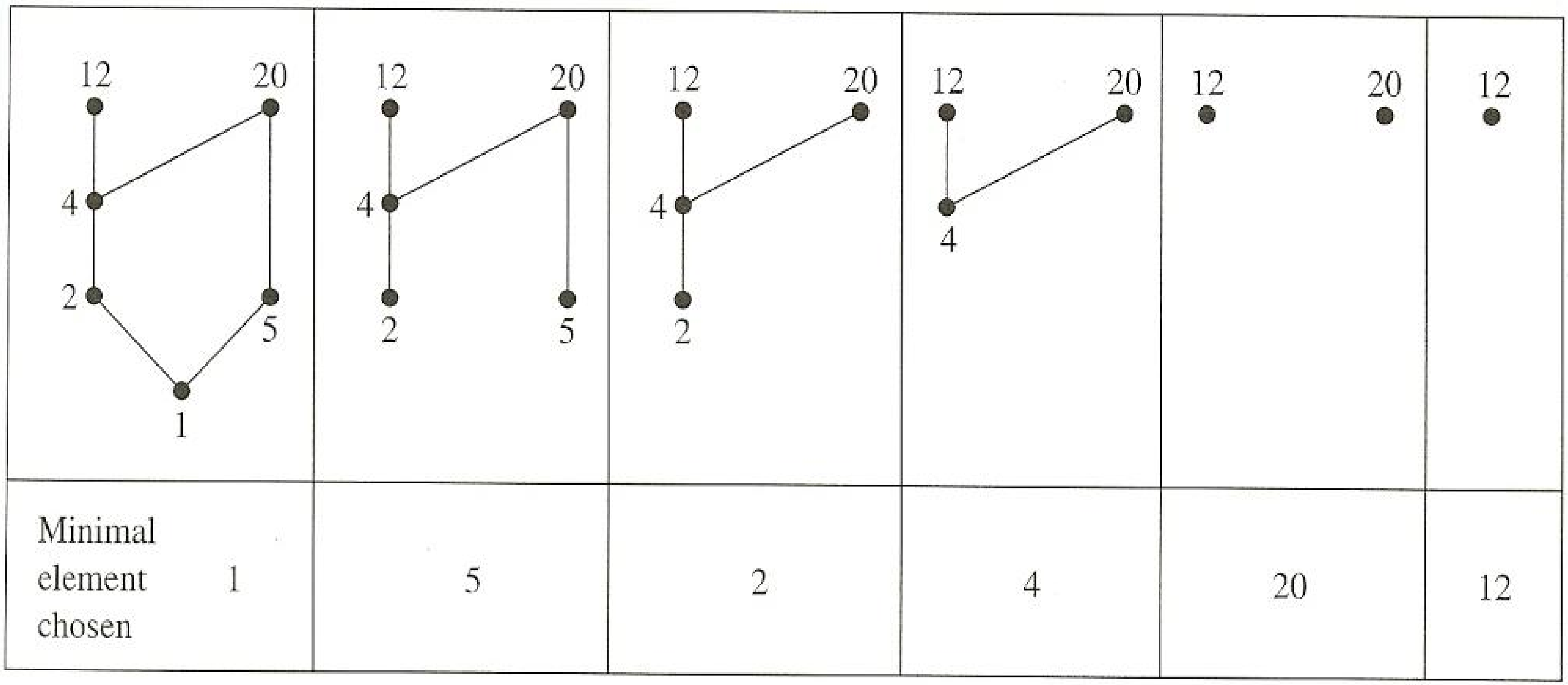
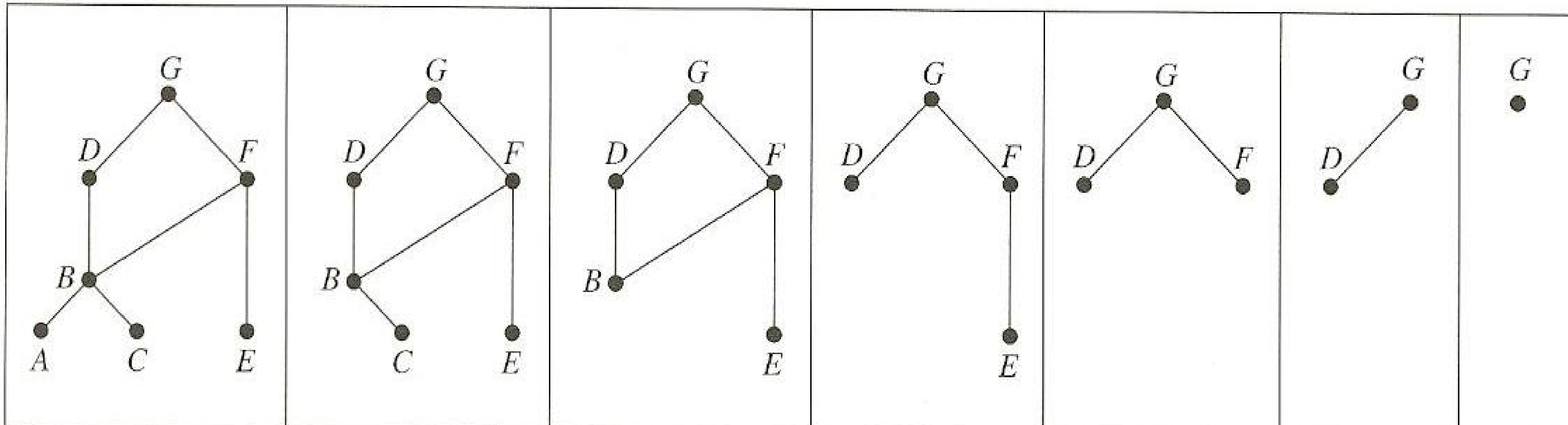


FIGURE 9 **A Topological Sort of ({1, 2, 4, 5, 12, 20}, |).**

Examples (2/2)



Minimal element chosen	*A*	*C*	*B*	*E*	*F*	*D*	*G*

FIGURE 11 **A Topological Sort of the Tasks.**