1

Logic Design

Chia-Tso Chao 趙家佐

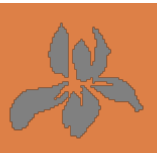
Dept. of Electronics Engineering

National Chiao Tung University

Welcome to Logic Design Class

- **Time/location：Tuesday CD and Thursday B @ ED 116**
- **Course webpage：http://tiger.ee.nctu.edu.tw/Course.php**
- **Office our：Thursday EF (made by appointment)**
- **Office：ED 625, Ext: 31671**
- **Text Book:**
  - C. H. Roth, Jr. and L. L. Kinney, Fundamentals of Logic Design, 7th edition, Cengage Learning, 2014.
- **Teaching Assistant: (Lab ED 612, ext. 54178)**
  - 張玟翔 , o0000032@yahoo.com.tw
  - 林建孝, c871111116.eecs99@nctu.edu.tw
  - 洪維澤, victor@ictek.net
- **Grading:**
  - Quiz 30% (14次小考 取 12次較高成績)
  - Midterm 35%
  - Final 35%

UNIT 1
NUMBER SYSTEMS AND CONVERSIONS



Spring 2016

Number Systems and Conversions

- **Contents**
  - Number systems and conversion
  - Binary arithmetic
  - Representation of negative numbers
    - Addition of two's complement numbers
    - Addition of one's complement numbers
  - Binary codes
- **Readings**
  - Unit 1.2~1.5

Number Systems (1/2)

- **Positional notation:** Each digit is multiplied by an appropriate power of base depending on its position in the number
  - The point separates the positive and negative powers of base
  - e.g., decimal (base 10) numbers
    - $953.78_{10} = 9x10^{2}+5x10^{1}+3x10^{0}+7x10^{-1}+8x10^{-2}$

      (210.-1-2): powers of base

  - A positive number N with base (radix) R (R positive integer, $R>1$):

    $N = (a_4a_3\,a_2a_1a_0.a_{-1}a_{-2}a_{-3})_R$

    $= a_4xR^4+a_3xR^3+a_2xR^2+a_1xR^1+a_0xR^0+a_{-1}xR^{-1}+a_{-2}xR^{-2}+a_{-3}xR^{-3}$
    - Base is also called radix
    - Base is indicated as subscript
  - Q: Why do people use the decimal number system?

Number Systems (2/2)

- **Examples:**
  - Decimal (base 10) numbers
    - $953.78_{10}=9x10^{2}+5x10^{1}+3x10^{0}+7x10^{-1}+8x10^{-2}$
  - Binary (base 2) numbers
    - $1011.11_{2}=1x2^{3}+0x2^{2}+1x2^{1}+1x2^{0}+1x2^{-1}+1x2^{-2}=11.75_{10}$
  - Octal (base 8) numbers
    - $147.3_{8}=1x8^{2}+4x8^{1}+7x8^{0}+3x8^{-1}=103.375_{10}$
  - Hexadecimal (base 16) numbers
    - $A2F_{16}=10x16^{2}+2x16^{1}+15x16^{0}=2607_{10}$

Common Number Systems

Name	Decimal	Binary	Octal	Hexadecimal
Radix	10	2	8	16
Digits	0,1,2,3,4, 5,6,7,8,9	0,1	0,1,2,3, 4,5,6,7	0,1,2,3,4,5,6,7, 8,9,A,B,C,D,E,F
First 17 positive integers	0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16	0 1 10 11 100 101 110 111 1000 1001 1010 1011 1100 1101 1110 1111 10000	0 1 2 3 4 5 6 7 10 11 12 13 14 15 16 17 20	0 1 2 3 4 5 6 7 8 9 A B C D E F 10

Conversion of a Decimal Integer

- **Convert a decimal integer to base *R* using division:**
  - $N = (a_n a_{n-1} \ldots a_2 a_1 a_0)_R = a_n R^n + a_{n-1} R^{n-1} + \ldots + a_2 R^2 + a_1 R^1 + a_0$
  - $N/R = a_n R^{n-1} + a_{n-1} R^{n-2} + \ldots + a_2 R^1 + a_1 = Q_1$ ..... remainder a_0
  - $Q_1/R = a_n R^{n-2} + a_{n-1} R^{n-3} + \ldots + a_3 R^1 + a_2 = Q_2$ ..... remainder a_1
  - $Q_2/R = a_n R^{n-3} + a_{n-1} R^{n-4} + \ldots + a_3 = Q_3$ ..... remainder a_2
  - …
  - Continued until we finally obtain ……remainder a_n
  - e.g., convert 53_{10} to binary

```
2 | 53
2 | 26     …… remainder = 1 = a0 (LSB)
2 | 13     …… remainder = 0 = a1
2 |  6     …… remainder = 1 = a2        53(10) = 110101(2)
2 |  3     …… remainder = 0 = a3
2 |  1     …… remainder = 1 = a4
     0     …… remainder = 1 = a5 (MSB)
```

Conversion of a Decimal Fraction (1/2)

- **Convert a decimal fraction to base *R* using multiplication:**
  - $F = (.a_{-1}a_{-2}a_{-3}...a_{-m})_R = a_{-1}R^{-1} + a_{-2}R^{-2} + a_{-3}R^{-3} + ... + a_{-m}R^{-m}$
  - $FR = a_{-1} + a_{-2}R^{-1} + a_{-3}R^{-2} + ... + a_{-m}R^{-m+1} = a_{-1} + F_1$
  - $F_1R = a_{-2} + a_{-3}R^{-1} + ... + a_{-m}R^{-m+2} = a_{-2} + F_2$
  - $F_2R = a_{-3} + ... + a_{-m}R^{-m+3} = a_{-3} + F_3$
  - …
  - Continued until we finally obtain a sufficient number of digits
  - e.g., convert $.625_{10}$ to binary

integer fraction

```
      .625
    x    2
-----------
 (1).250   (a-1=1, MSB)
    x    2
-----------
 (0).500   (a-2=0)
    x    2
-----------
 (1).000   (a-3=1, LSB)
```

$.625_{10} = .101_2$

Conversion of a Decimal Fraction (2/2)

- **Sometimes, the result is a repeating fraction**
  - e.g., convert 0.7_{10} to binary

```
    .7
  x  2
-------
 (1).4
  x  2
-------
 (0).8
  x  2
-------
 (1).6
  x  2
-------
 (1).2
  x  2
-------
 (0).4
  x  2
-------
 (0).8
```

Start repeating!
.4 was previously obtained

0.7_{10} = .1 0110 0110 0110 …$_2$

Conversion between Two Bases (1/2)

- **Convert between two bases R_1 and R_2 other than decimal**
  - Base R_1 ⇒ base 10 ⇒ base R_2
  - e.g., convert 231.3_4 to base 7

$$231.3_4 = 2\times4^2+3\times4^1+1\times4^0+3\times4^{-1} = 45.75_{10}$$

integer

```
7 | 45
7 |  6   …… rem. 3
    0   …… rem. 6
⇒ 63₇
```

fraction

```
   .75
 x   7
(5).25
 x   7
(1).75
 x   7
(5).25
 x   7
(1).75
```

Repeat! ⇒ $.\underline{51}_7$

integer fraction

$$231.3_4 = 45.75_{10} = 63.\underline{51}_7$$

Conversion between Two Bases (2/2)

- **Convert between binary and octal/hexadecimal by inspection**
  1. Start at the binary point
  2. Divide bits into groups of three/four, adding 0's if necessary
  3. Replace each group by an octal/hexadecimal digit
  4. And vice versa
  - Binary ⇔ octal

    $1001101.010111_2 = \underline{001}\ \underline{001}\ \underline{101} . \underline{010}\ \underline{111}_2 = 115.27_8$

    1 1 5 2 7

    adding 0's

  - Binary ⇔ hexadecimal

    $1001101.010111_2 = \underline{0100}\ \underline{1101} . \underline{0101}\ \underline{1100}_2 = 4D.5C_{16}$

    4 D 5 C

Binary Arithmetic

Binary Arithmetic -- Addition

- **Addition table:**
  - 0 + 0 = 0
  - 0 + 1 = 1
  - 1 + 0 = 1
  - 1 + 1 = 0 (and carry 1 to the next column)
  - e.g., add 13_{10} and 11_{10} in binary

```
          1111  <-- carries
13₁₀ =    1101
11₁₀ =    1011
         -----
         11000 = 24₁₀
```

Binary Arithmetic -- Subtraction

- **Subtraction table:**
  - 0 – 0 = 0
  - 1 – 0 = 1
  - 1 – 1 = 0
  - 0 – 1 = 1 (and borrow 1 from the next column)
    - Borrow 1 from the next column
      ≡ have -1 at the next column
      = obtain *R* at the current column
  - e.g., subtract 19_{10} from 29_{10} in binary

```
              1 <-------- borrow
29(10) =  11101
19(10) =  10011
          ------
          1010 = 10(10)
```

Binary Arithmetic -- Multiplication

- **Multiplication table:**
  - 0 x 0 = 0
  - 0 x 1 = 0
  - 1 x 0 = 0
  - 1 x 1 = 1
  - e.g., multiply 13_{10} by 11_{10} in binary

```
      1101 = 13₁₀ multiplicand
x     1011 = 11₁₀ multiplier
-----------
      1101
     1101   <-- copy of multiplicand if "1"
    0000
   1101
-----------
 10001111 = 143₁₀
```

Binary Arithmetic -- Division

- **Is similar to (easier than) decimal division**
  - e.g., divide 145_{10} by 11_{10} in binary

```
                        1101 = quotient
divisor 11₁₀ = 1011 | 10010001 = 145₁₀ dividend
                      1011
                       1110
                       1011
                        1101
                        1011
                          10 = remainder
```

Representation of Negative Numbers

Addition of two's complement numbers

Addition of one's complement numbers

Negative Numbers

- **Sign and Magnitude: 1-bit sign + (*n*-1)-bit magnitude**

  - e.g., $0011_{sm} = +3_{10}$, $1011_{sm} = -3_{10}$
  - Is common for people but awkward for computers
- **1's complement: $N\_ = (2^n - 1) - N$**
  - Complement +*N* bit-by-bit
  - e.g., +3 = 0011, 3_ = 1100
- **2's complement: $N^* = 2^n - N = N\_ + 1$**
  - Complement +*N* bit-by-bit and then add 1
  - Or complement all bits to the left of the rightmost 1
  - e.g., +3 = 0011, 3* = 1101

Signed Binary Integers

Word length: $n = 4$

+N	Positive integers (all systems)	-N	Negative integers		
			Sign and magnitude	2's complement N*	1's complement N_
+0	0000	-0	1000	----	1111
+1	0001	-1	1001	1111	1110
+2	0010	-2	1010	1110	1101
+3	0011	-3	1011	1101	1100
+4	0100	-4	1100	1100	1011
+5	0101	-5	1101	1011	1010
+6	0110	-6	1110	1010	1001
+7	0111	-7	1111	1001	1000
		-8	----	1000	----

- **For word length *n*, there are 2^n different permutations**
  - SM and *N*_: [+7, …, +0, -0, …, -7] (redundant -0)
  - *N**: [+7, …, +0, …, -8] ($-2^{n-1} \sim +2^{n-1}-1$)
- **View the first bit as the sign bit: positive/negative ⇔ 0/1**

Warning: Unsigned or Signed Numbers

- **For word length *n*, there are 2^n different permutations**
  - Unsigned:
    - $[0, 2^n-1]$
  - Signed
    - SM (sign and magnitude) and *N*_ (1's complement) : $[-2^{n-1}+1, +2^{n-1}-1]$ (redundant -0)
    - *N** (2's complement) : $[-2^{n-1}, +2^{n-1}-1]$
- **e.g., what is 1110?**

Addition of Two's Complement Numbers

1. **Add just as if all numbers were positive**
2. **Discard the carry from the sign bit**
   - Why to discard the carry? (i.e., subtract 2^n)
     1. Add(-A, +B), $B > A$
        - $A^* + B = (2^n - A) + B = 2^n + (B - A)$
        - $\Rightarrow -2^n \Rightarrow (B - A)$
     2. Add(-A, -B), $A+B \leq 2^{n-1}$
        - $A^* + B^* = (2^n - A) + (2^n - B) = 2^n + 2^n - (A + B) = 2^n + (A + B)^*$
        - $\Rightarrow -2^n \Rightarrow (A + B)^*$
   - **e.g.,**

Add(+A, +B) $A+B < 2^{n-1}$		Add(+A, -B) $B > A$		Add(-A, +B) $B > A$		Add(-A, -B) $A+B \leq 2^{n-1}$	
+3	0011	+5	0101	-5	1011	-3	1101
+4	0100	-6	1010	+6	0110	-4	1100
+7	0111	-1	1111	+1	(1)0001	-7	(1)1001

Exception: Overflow in Two's Complement

- **An overflow means out of range of representation**
  - When the word length is *n* bits, the correct representation of a number (including sign) requires more than *n* bits
- **How to detect overflow?**
  - Check sign!
    - (+) + (+) ⇒ (-) or (-) + (-) ⇒ (+)
- **e.g.,**

Add(+*A*, +*B*) ***A*+*B* ≥ 2^{n-1}**	**Add(-*A*, -*B*)** ***A*+*B* > 2^{n-1}**
+5 0101 +6 0110 1011 Wrong answer! Overflow: +11 requires 5 bits including sign	-5 1011 -6 1010 (1)0101 Wrong answer! Overflow: -11 requires 5 bits including sign

Addition of One's Complement Numbers

- **End-around carry: Add the carry out back to the rightmost bit**
  - Why to take end-around carry? (i.e., subtract 2^n & add 1)
    1. Add(-*A*, +*B*), *B* > *A*
       - $A\_ + B = (2^n - 1 - A) + B = 2^n + (B - A) - 1$
       - $\Rightarrow -2^n \,\&\, +1 \Rightarrow (B - A)$
    2. Add(-*A*, -*B*), $A+B \le 2^{n-1}$
       - $A\_ + B\_ = (2^n - 1 - A) + (2^n - 1 - B) = 2^n + 2^n - 1 - (A + B) - 1$
         $= 2^n + (A + B)\_ - 1$
       - $\Rightarrow -2^n \,\&\, +1 \Rightarrow (A + B)\_$
  - **e.g.,**

Add(+*A*, +*B*) $A+B < 2^{n-1}$		Add(+*A*, -*B*) *B* > *A*		Add(-*A*, +*B*) *B* > *A*		Add(-*A*, -*B*) $A+B \le 2^{n-1}$	
+3	0011	+5	0101	-5	1010	-3	1100
+4	0100	-6	1001	+6	0110	-4	1011
+7	0111	-1	1110	+1	(1)0000	-7	(1)0111
					1		1
					0001		1000

Exception: Overflow in One's Complement

- **How to detect overflow?**
  - Check sign!
    - (+) + (+) ⇒ (-) or (-) + (-) ⇒ (+)
- **e.g.,**

Add(+*A*, +*B*) $A+B \geq 2^{n-1}$	Add(-*A*, -*B*) $A+B \geq 2^{n-1}$
+5 0101 +6 0110 1011 Wrong answer! Overflow: +11 requires 5 bits including sign	-5 1010 -6 1001 (1)0011 1 0100 Wrong answer! Overflow: -11 requires 5 bits including sign

Binary Codes

Decimal Digits to Binary Codes

- **Input/output interface generally uses decimal numbers**
- **How to represent decimal numbers using binary codes?**
  - Choose 10 elements from 16 binary numbers of 4 bits
  - e.g., 937.25 ⇒ BCD: 1001 0011 0111 . 0010 0101

Many options!

Decimal digit	8-4-2-1 code	6-3-1-1 code	Excess-3 code	2-out-of-5 code	Gray code
0	0000	0000	0011	00011	0000
1	0001	0001	0100	00101	0001
2	0010	0011	0101	00110	0011
3	0011	0100	0110	01001	0010
4	0100	0101	0111	01010	0110
5	0101	0111	1000	01100	0111
6	0110	1000	1001	10001	0101
7	0111	1001	1010	10010	0100
8	1000	1011	1011	10100	1100
9	1001	1100	1100	11000	1101
Property	Weighted a.k.a. BCD	Weighted	= BCD+3	Only 2 bits are one ⇒ For error checking	Only 1 bit changes as counting up/down ⇒ For analog quantities

Warning: Conversion or Coding?

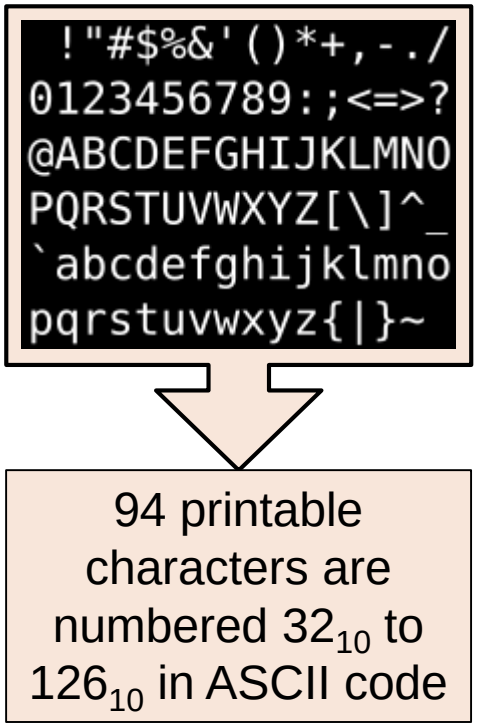
- **Do NOT mix up conversion of a decimal number to a binary number with coding a decimal number with a binary code**
  - e.g.,

    $13_{10} \Leftrightarrow 1101_2$ (This is conversion)

    $13 \Leftrightarrow$ 0001 0011 (This is coding)

Text to Binary Codes

- **ASCII**
  - American Standard Code for Information Interchange
  - English alphanumeric symbols
  - 7 bits
- **Big-5 code**
  - Traditional Chinese characters
  - Double-byte coding (16 bits)



Homework of Unit 1

- **1.4, 1.5, 1.7, 1.19, 1.28, 1.39, 1.44**