UNIT 5
KARNAUGH MAPS

Spring 2011

Karnaugh Maps

- **Contents**
  - Minimum forms of switching functions
  - Two- and three-variable Karnaugh maps
  - Four-variable Karnaugh maps
  - Determination of minimum expressions using essential prime implicants
  - Five-variable Karnaugh maps
  - Other forms of Karnaugh maps
  - Other uses of Karnaugh maps
- **Reading**
  - Unit 5

Recap: Logic Design

- **Design a combinational logic circuit starting with a word description of the desired circuit behavior**
- **Steps:**
  1. Translate the word description into a switching function (Unit 4)
     - Truth table
     - Boolean expression: SOP/POS derived from minterm/maxterm expansion (Unit 4)
  2. Simplify the function
     - Boolean algebra (Units 2&3)
     - Karnaugh map (Unit 5)
     - Quine-McCluskey (Unit 6)
     - … etc
  3. Realize it using available logic gates

Difficulties in Algebraic Simplification

- **Problems:**
  - Difficult to apply in a systematic way
  - Difficult to tell when you have arrived at a minimum solution
    - Minimum SOP/POS
      1. Minimum # of terms (i.e., # of gates)
      2. Minimum # of literals (i.e., # of gate inputs)
- **Solutions: systematic methods**
  1. Karnaugh map (K-map) (Unit 5)
     - Especially useful for 3 or 4 variables
     - Faster and easier than other methods
  2. Quine-McCluskey (Unit 6)
  3. … etc.

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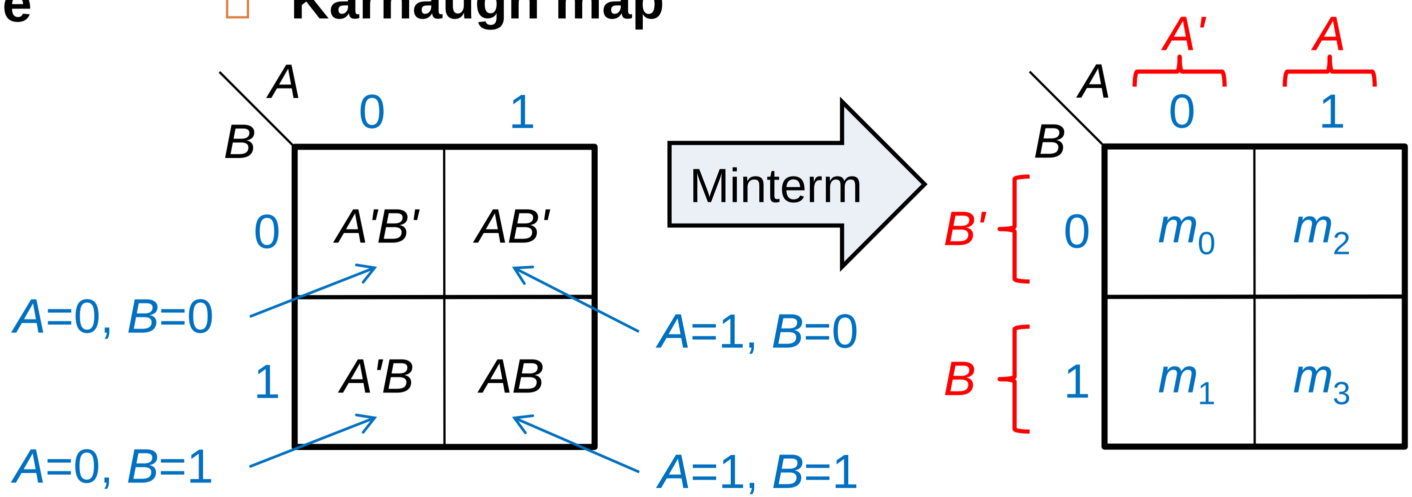
Two- & Three-Variable Karnaugh Maps

Two-Variable Karnaugh Maps (1/2)

- **Truth table = minterm expansion = Karnaugh map**
  - Each square of the K-map corresponds to a combination of values of inputs
  - i.e., each square = a minterm = a row in truth table
- **Truth table**

m_i	A	B	F
0	0	0	a_0
1	0	1	a_1
2	1	0	a_2
3	1	1	a_3

- **Karnaugh map**



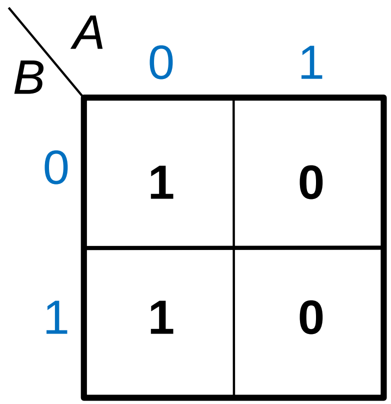
Two-Variable Karnaugh Maps (2/2)

- **e.g.,**

1. Truth table

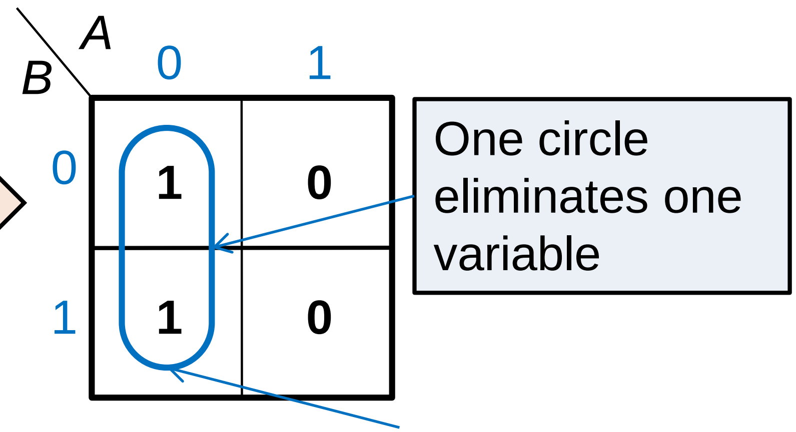
m_i	A	B	F
0	0	0	1
1	0	1	1
2	1	0	0
3	1	1	0

2. K-map



3. Simplification in K-map

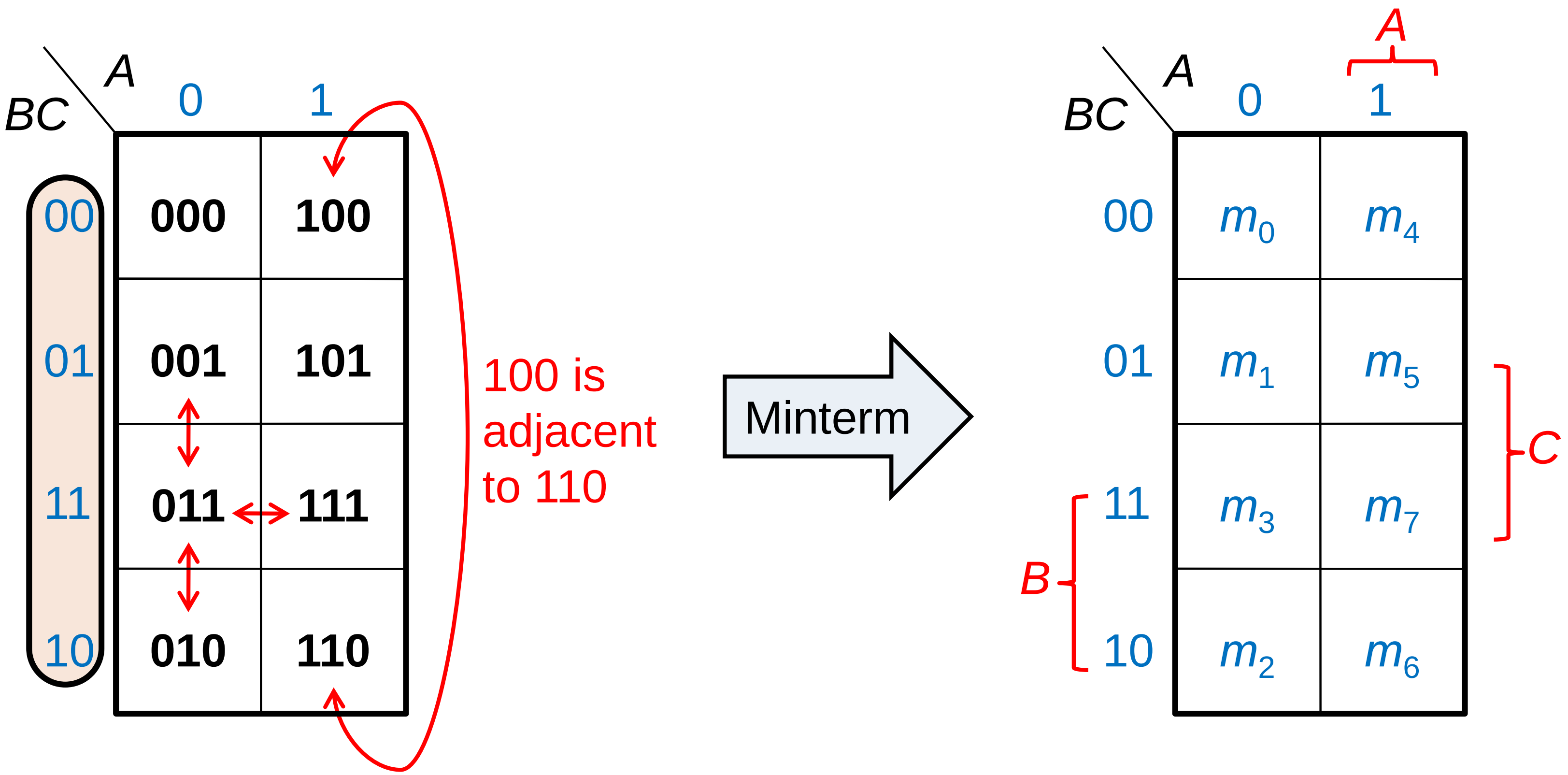
$XY'+XY = X(Y'+Y) = X$



$F = A'B'+A'B = A'(B'+B) = A'$

Three-Variable Karnaugh Maps

- **Minterms in adjacent squares of K-map differ in only ONE bit**
- **⇒ Combine them, $XY'+XY = X(Y'+Y) = X$**



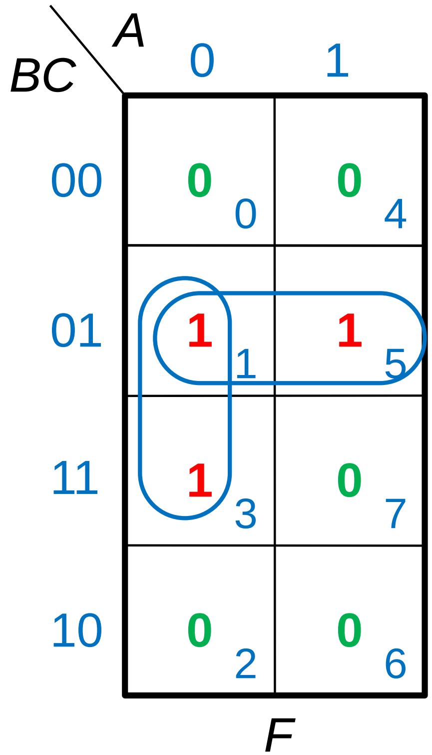
Example: $F(A, B, C) = \Sigma\, m(1, 3, 5)$

- **e.g., $F(A, B, C) = \Sigma\, m(1, 3, 5) = \Pi\, M(0, 2, 4, 6, 7)$**

1. Truth table

m_i	A	B	C	F
0	0	0	0	0
1	0	0	1	1
2	0	1	0	0
3	0	1	1	1
4	1	0	0	0
5	1	0	1	1
6	1	1	0	0
7	1	1	1	0

2. K-map



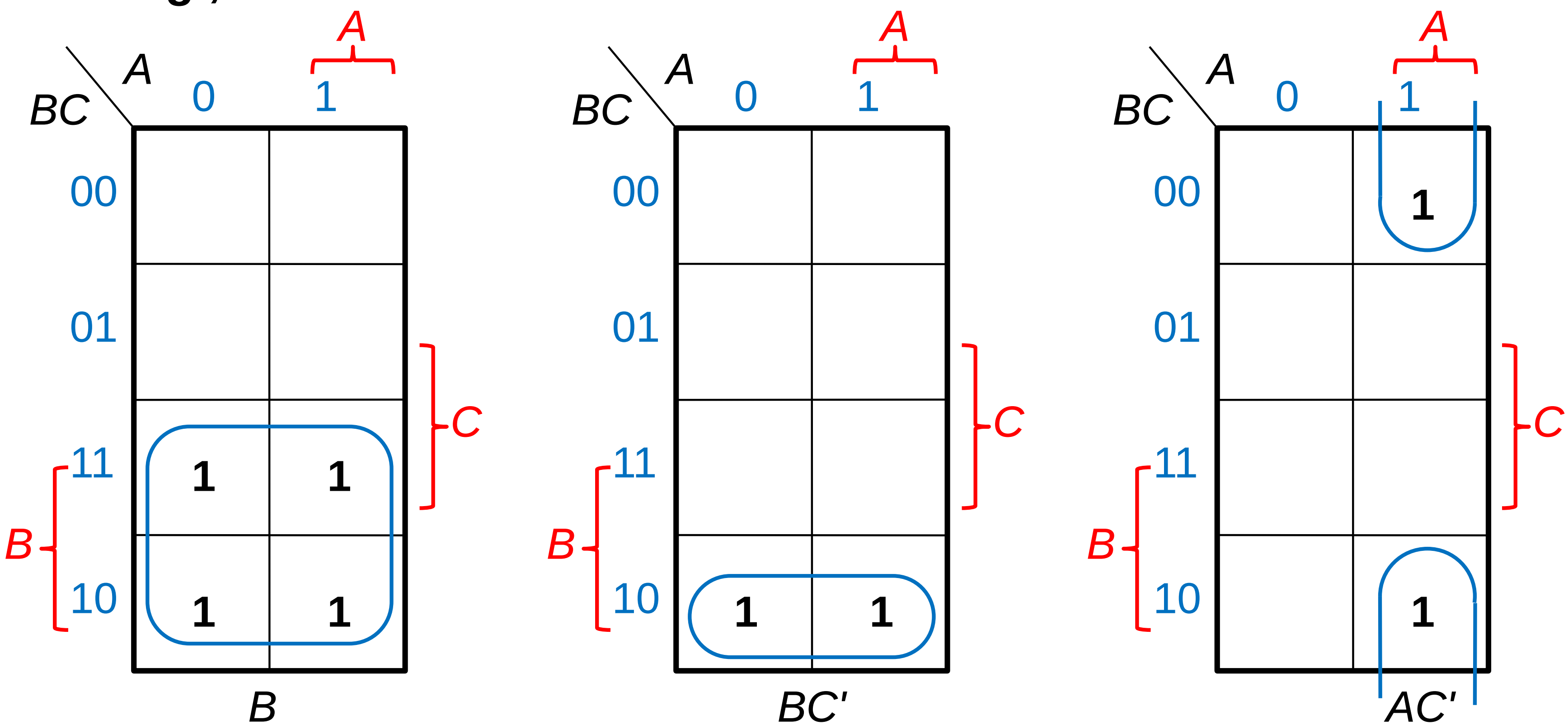
3. Simplification in K-map

$F = A'B'C + A'BC + AB'C = A'C + B'C$

$\Rightarrow$ Minimum SOP form

Product Terms in Karnaugh Maps

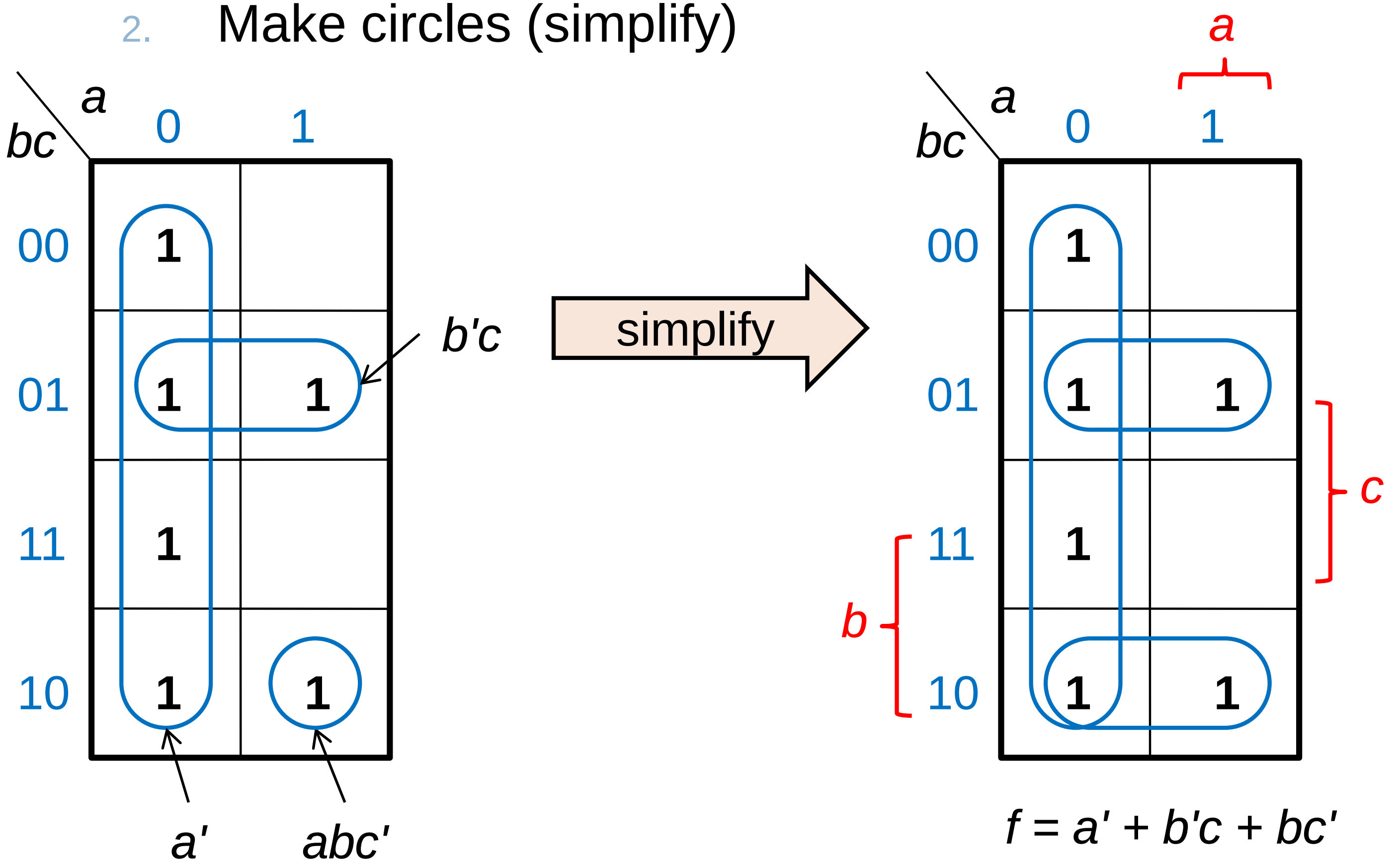
- **e.g.,**



($A'BC + ABC + A'BC' + ABC' = B$)

Example: $f(a, b, c) = abc' + b'c + a'$

- **e.g., $f(a, b, c) = abc' + b'c + a'$**
  1. Mark 1's
  2. Make circles (simplify)



$f = a' + b'c + bc'$

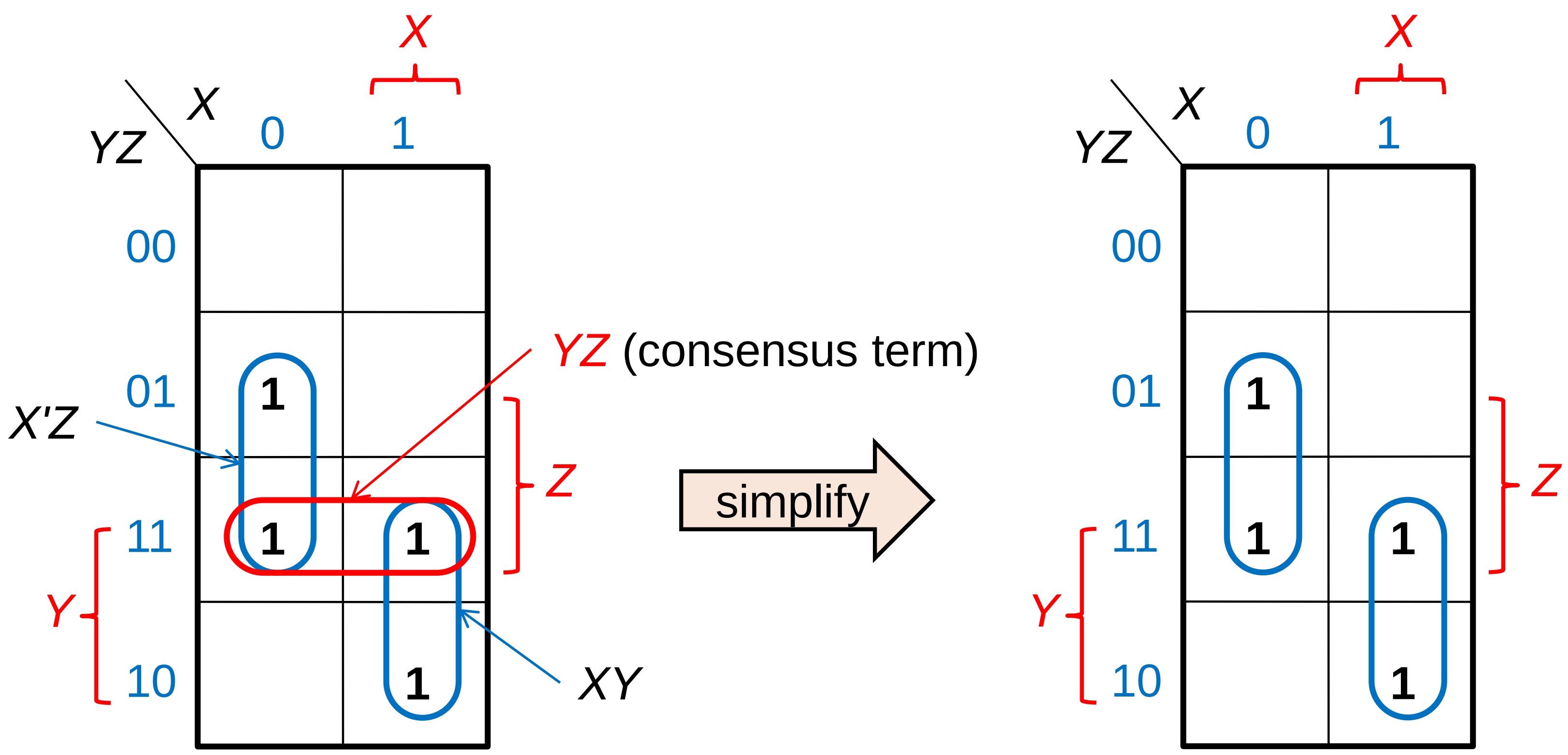
$X + X = X$

Make the circle as large as possible

Karnaugh maps

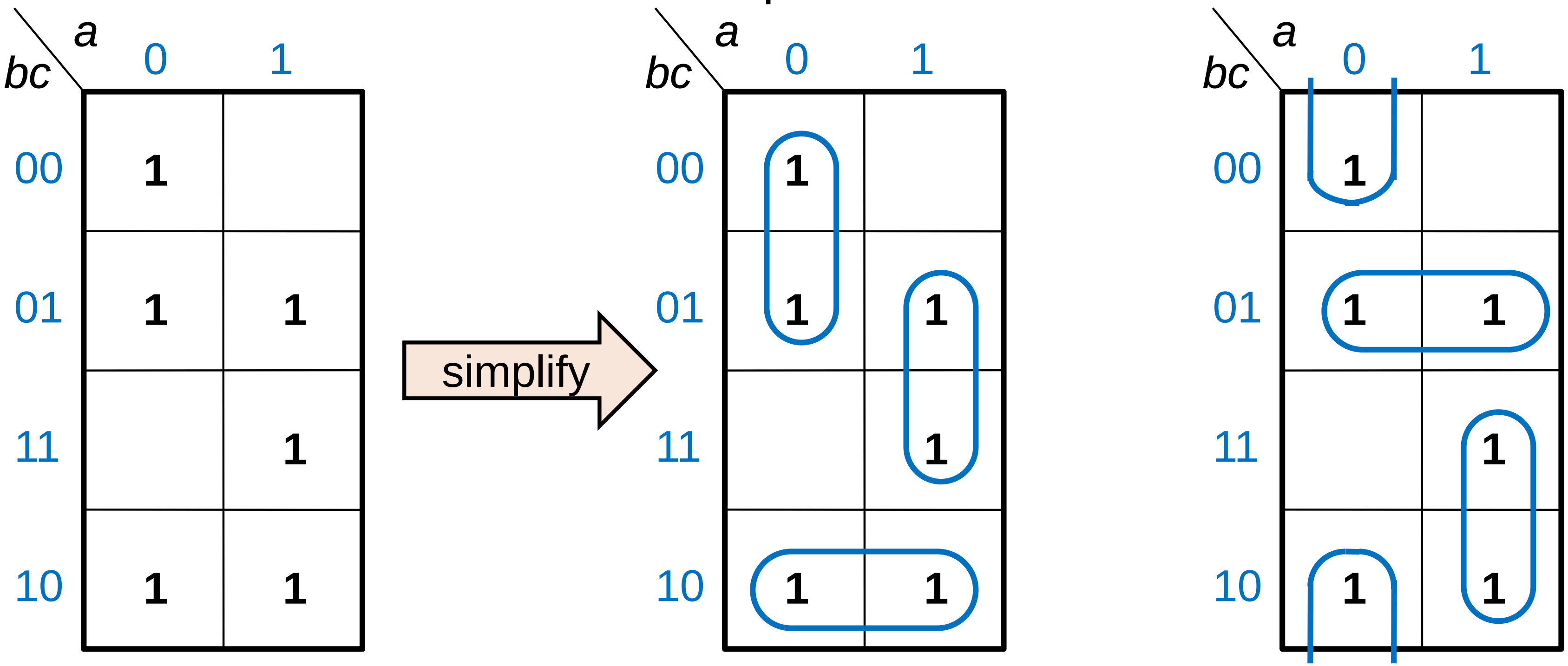
The Consensus Theorem in K-Map

- **Overlapped circles imply redundant terms**
- **e.g., the consensus theorem**
  - $XY + X'Z + YZ = XY + X'Z$ (YZ is redundant)



All Solutions Are Shown in Karnaugh Maps

- **All possible minimum SOPs can be determined from K-map**
  - # of terms and # of literals
- **e.g.,** $\boldsymbol{F = \Sigma\, m(0, 1, 2, 5, 6, 7)}$
  - Make each circle as large as possible
  - Select as few circles as possible to cover all minterms



Karnaugh maps

1. $F = a'b' + bc' + ac$

2. $F = a'c' + b'c + ab$

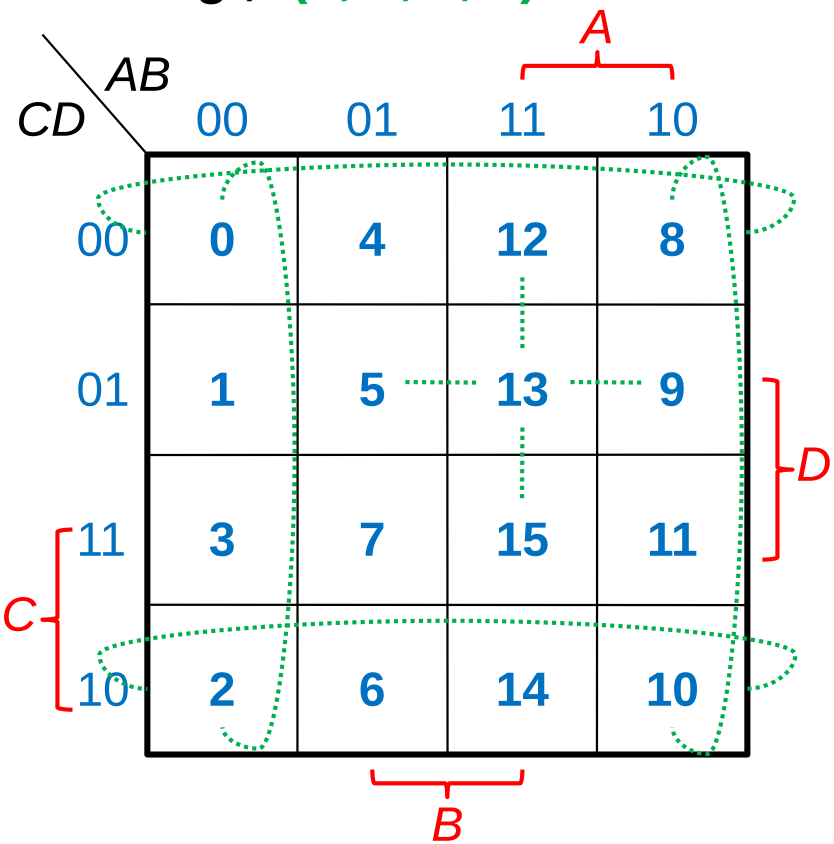
Summary

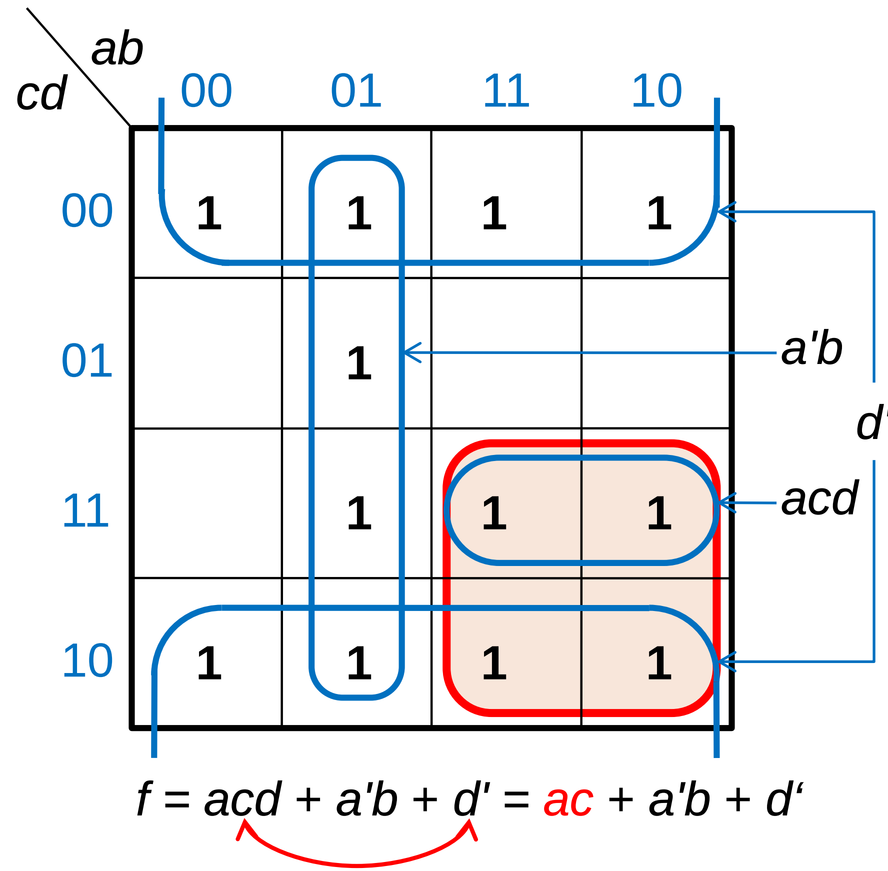
- **Truth table = minterm expansion = Karnaugh map**
- **Simplification in Karnaugh maps**
  - Apply $XY'+XY = X(Y'+Y) = X$ and $X + X = X$
  - Minimum SOP = (min # of terms, min # of literals)
  - Steps: (Adjacent squares differ in only one bit)
    1. Mark 1's
    2. Make circles
       - Make each circle as large as possible (# of literals)
       - Select as few circles as possible to cover all 1's (# of terms)
- **Algebraic simplification also holds in Karnaugh maps**
  - Rule A: combining terms: $XY + XY' = X$
  - Rule B: eliminating terms: $X + XY = X$; $XY + X'Z + YZ = XY + X'Z$
  - Rule C: eliminating literals: $X + X'Y = X + Y$
  - Rule D: adding redundant terms:
    $XY + X'Z = XY + X'Z + YZ$; $X = X + XY$

Four-Variable Karnaugh Maps

Four-Variable Karnaugh Maps

- **Adjacent squares differ in only one bit**
- **e.g.,** ***f(a, b, c, d) = acd + a'b + d'***



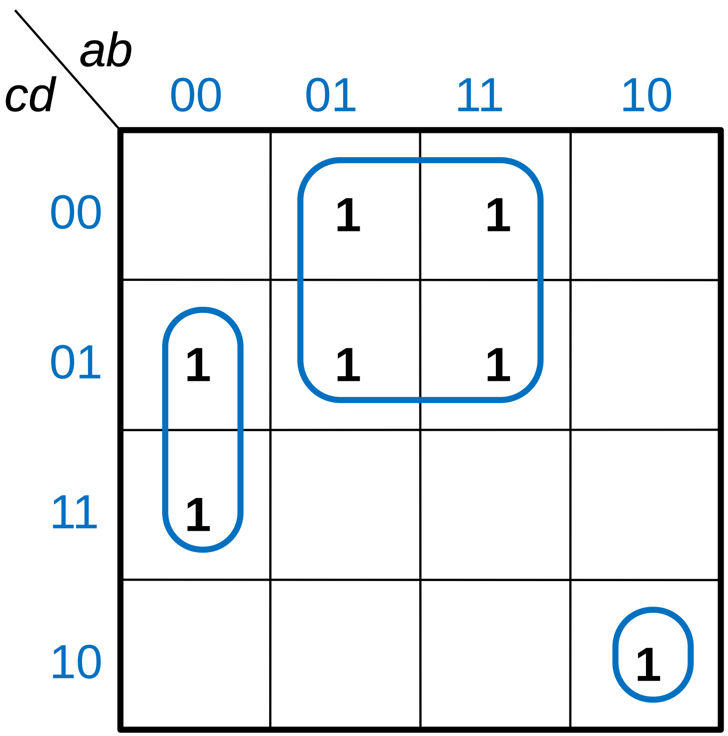


Make the circle as large as possible

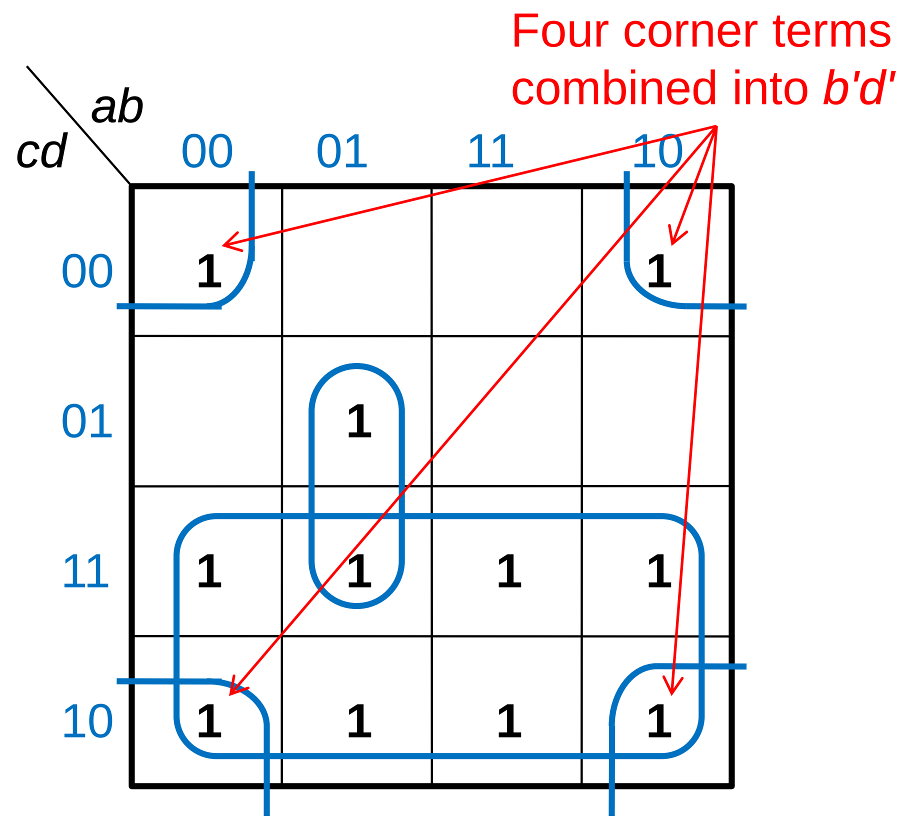
Two More Examples

$f_1 = \Sigma\, m(1,3,4,5,10,12,13)$

$f_2 = \Sigma\, m(0,2,3,5,6,7,8,10,11,14,15)$



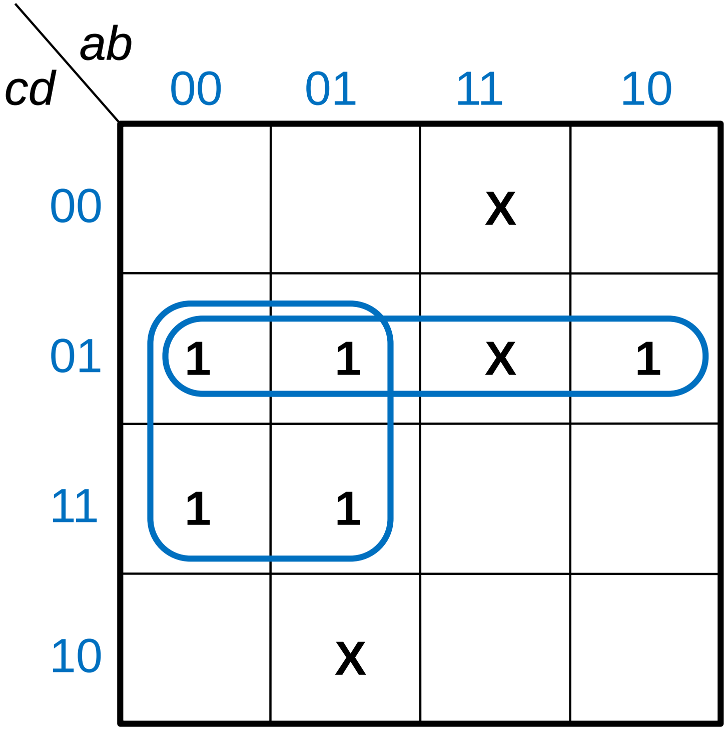
$f_1 = bc' + a'b'd + ab'cd'$



$f_2 = c + b'd' + a'bd$

Karnaugh Maps with Don't Cares

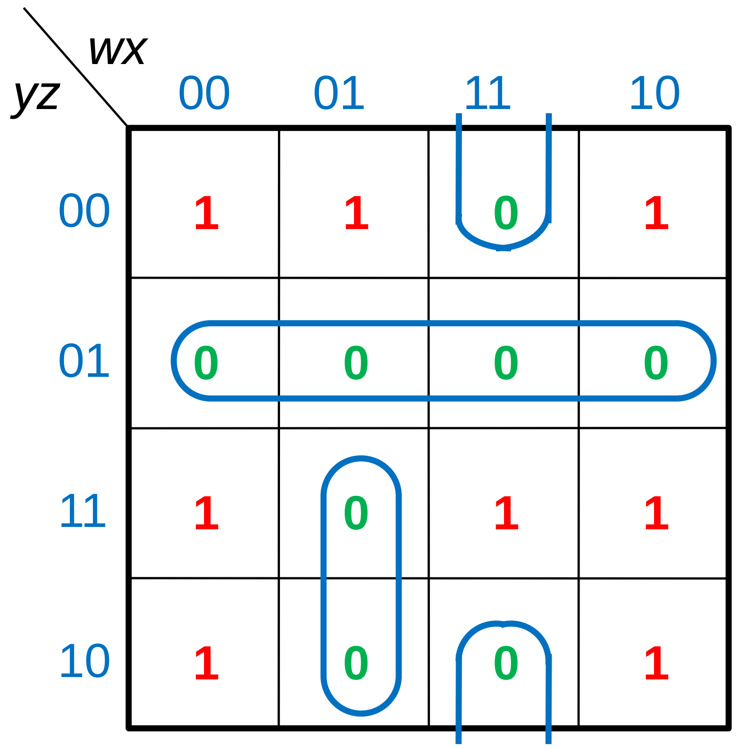
- **Don't cares can be assigned with 0's or 1's**
- **After assignment, the function becomes completely specified**
- **e.g., $f(a, b, c, d) = \Sigma\, m(1, 3, 5, 7, 9) + \Sigma\, d(6, 12, 13)$**



$f = a'd + c'd = \Sigma\, m(1, 3, 5, 7, 9, 13)$

Minimum POS?

- **Minimum SOP = circle 1's of *f***
- **Minimum POS = circle 0's of *f'***
  - Find min. SOP of *f'*, then complement it
- **e.g., *f = x'z' + wyz + w'y'z' + x'y***



$f' = y'z + w'xy + wxz'$

By DeMorgan's law,

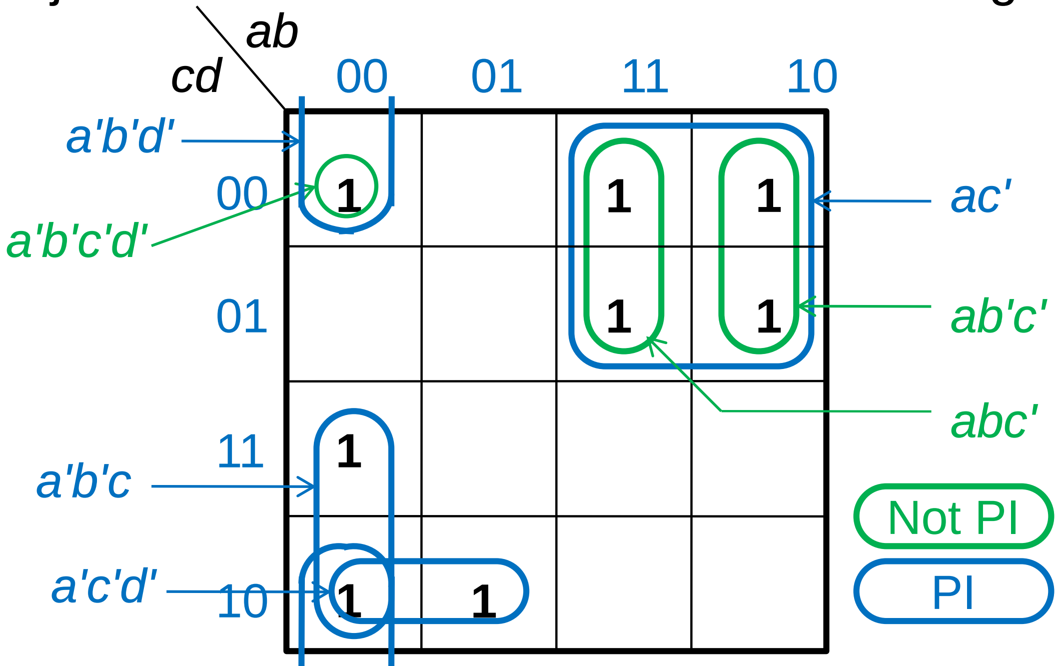
$f = (y'z + w'xy + wxz)'$

$= (y + z')(w + x' + y')(w' + x' + z)$

Prime Implicants

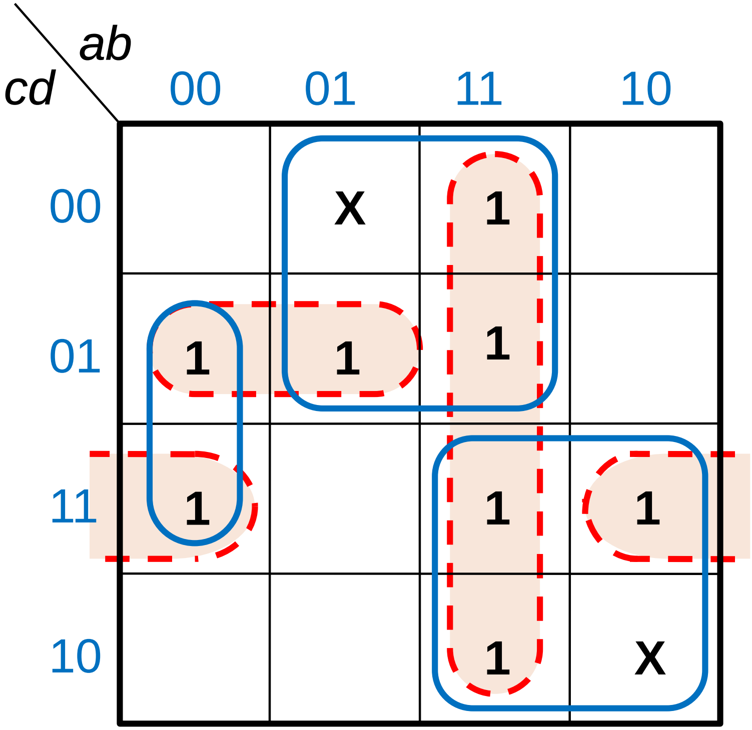
Prime Implicants (1/2)

- **Implicant: a product term**
  - i.e., any single 1 or any group of 1's in the K-map
- **Prime implicant (PI): an implicant that cannot be covered by other implicants**
  - i.e., a circle that cannot be enlarged any more
  - A single 1 is a PI if not adjacent to any other 1's
  - Two adjacent 1's form a PI if not contained in a group of 4 1's



Prime Implicants (2/2)

- **Cover: a set of prime implicants which covers all 1's**
- **A minimum SOP contains only prime implicants (Why?)**
  - ⇒ Minimum cover = (min # of PIs, min # of literals)
- **Don't cares are treated just like 1's**
- **e.g.,**



PI: *a'b'd*, *bc'*, *ac*, *a'c'd*, *ab*, *b'cd*

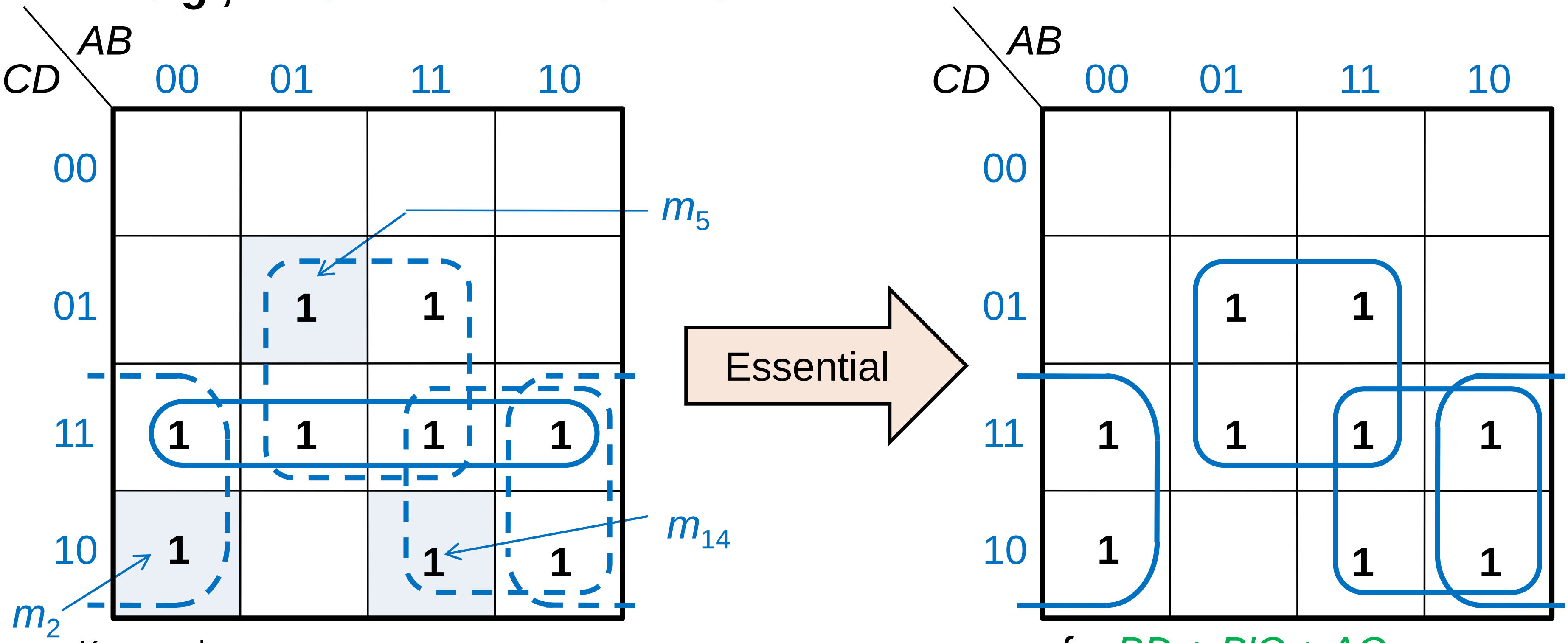
Min SOP:

1. *a'b'd* + *bc'* + *ac*
2. *a'c'd* + *ab* + *b'cd*

⇒ 1. is better

Essential Prime Implicants

- **Essential prime implicant: If a minterm is covered by only one PI, the PI is essential**
  - Essential PI MUST be included in minimum SOP
  - Find essential PI's = find the 1's circled only once
- **e.g., $f = CD + BD + B'C + AC$**

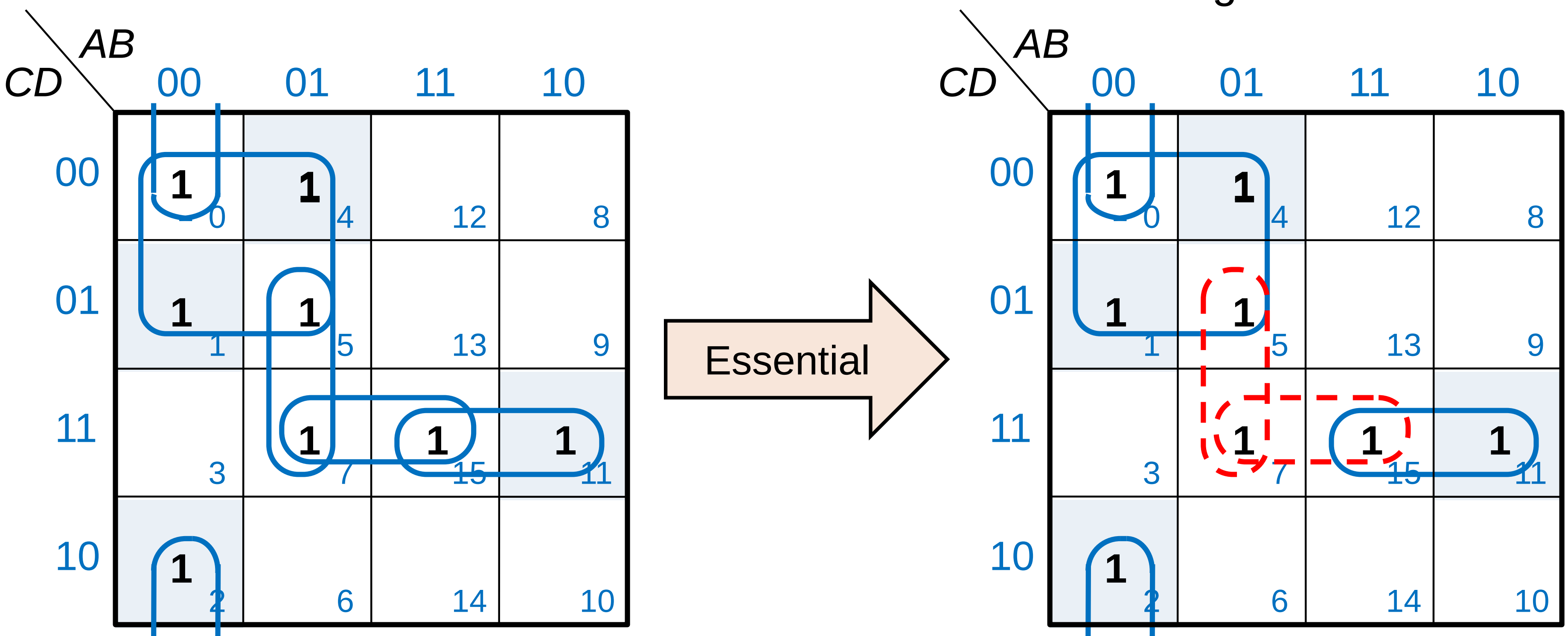


Karnaugh maps

$f = BD + B'C + AC$

One More General Example

- **Find minimum cover:**
  1. Find all PI's
  2. Find essential PI's
  3. Find a minimum set of PI's to cover the remaining 1's



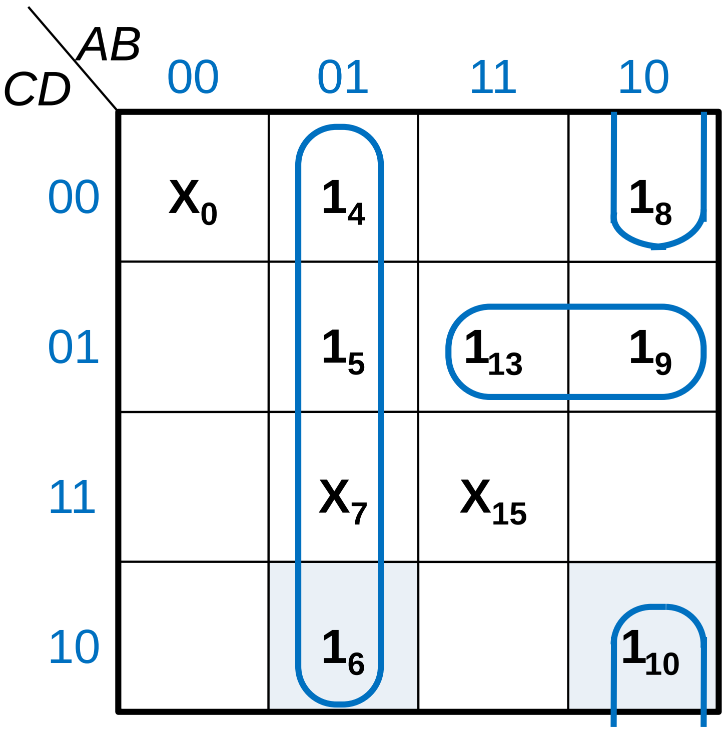
$$f = A'C' + A'B'D' + ACD + \begin{cases} A'BD \\ BCD \end{cases}$$

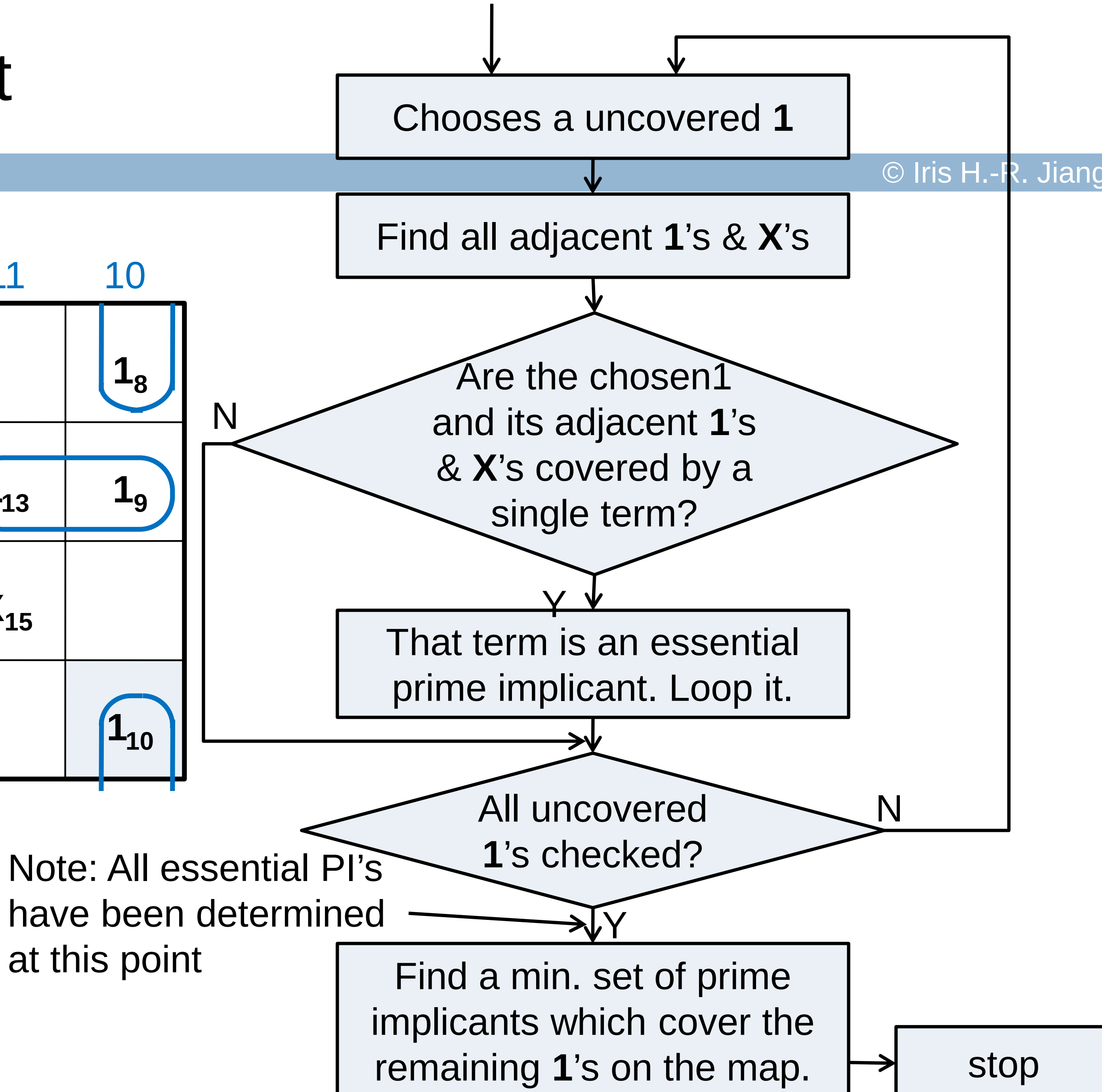
Karnaugh maps

Summary

- **Minimum SOP = minimum cover = a minimum set of PI's which cover all 1's**
  - Minimum cover = (min # of PIs, min # of literals)
- **Steps:**
  1. Find all PI's
  2. Find essential PI's
  3. Find a minimum set of PI's to cover the remaining 1's
- **Recap: steps of simplification in Karnaugh maps**
  1. Mark 1's
  2. Make circles
     - Make each circle as large as possible = find PI
     - Select as few circles as possible to cover all 1's = find min cover

Flowchart

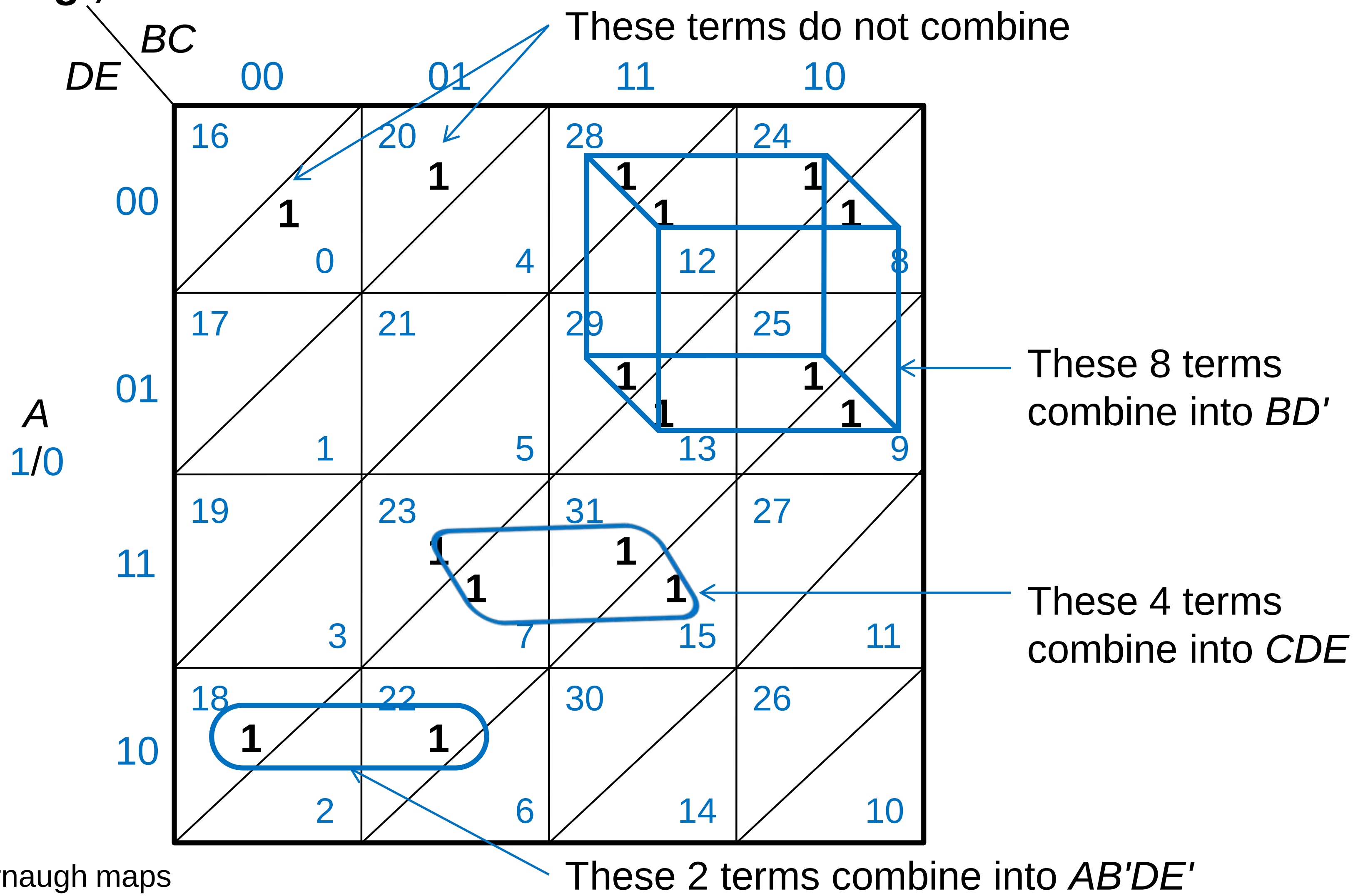

AB
CD
00
01
11
10
00
01
11
10
X_0
1_4
1_8
1_5
1_{13}
1_9
X_7
X_{15}
1_6
1_{10}


Chooses a uncovered **1**
Find all adjacent **1**'s & **X**'s
Are the chosen1 and its adjacent **1**'s & **X**'s covered by a single term?
N
Y
That term is an essential prime implicant. Loop it.
All uncovered **1**'s checked?
N
Y
Note: All essential PI's have been determined at this point
Find a min. set of prime implicants which cover the remaining **1**'s on the map.
stop

Five-Variable Karnaugh Maps

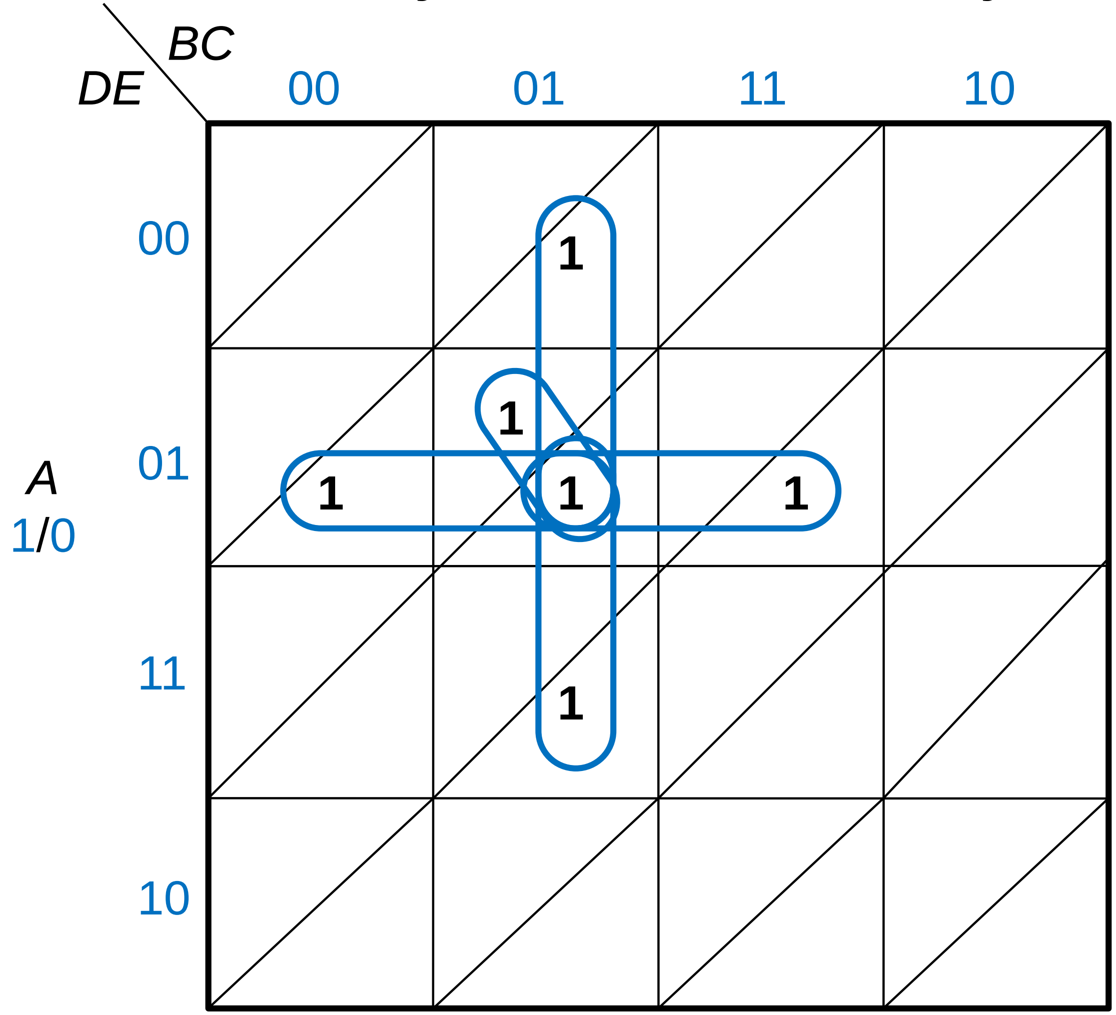
Five-Variable Karnaugh Maps

- **e.g.,**



Adjacency in 5-Variable Karnaugh Maps

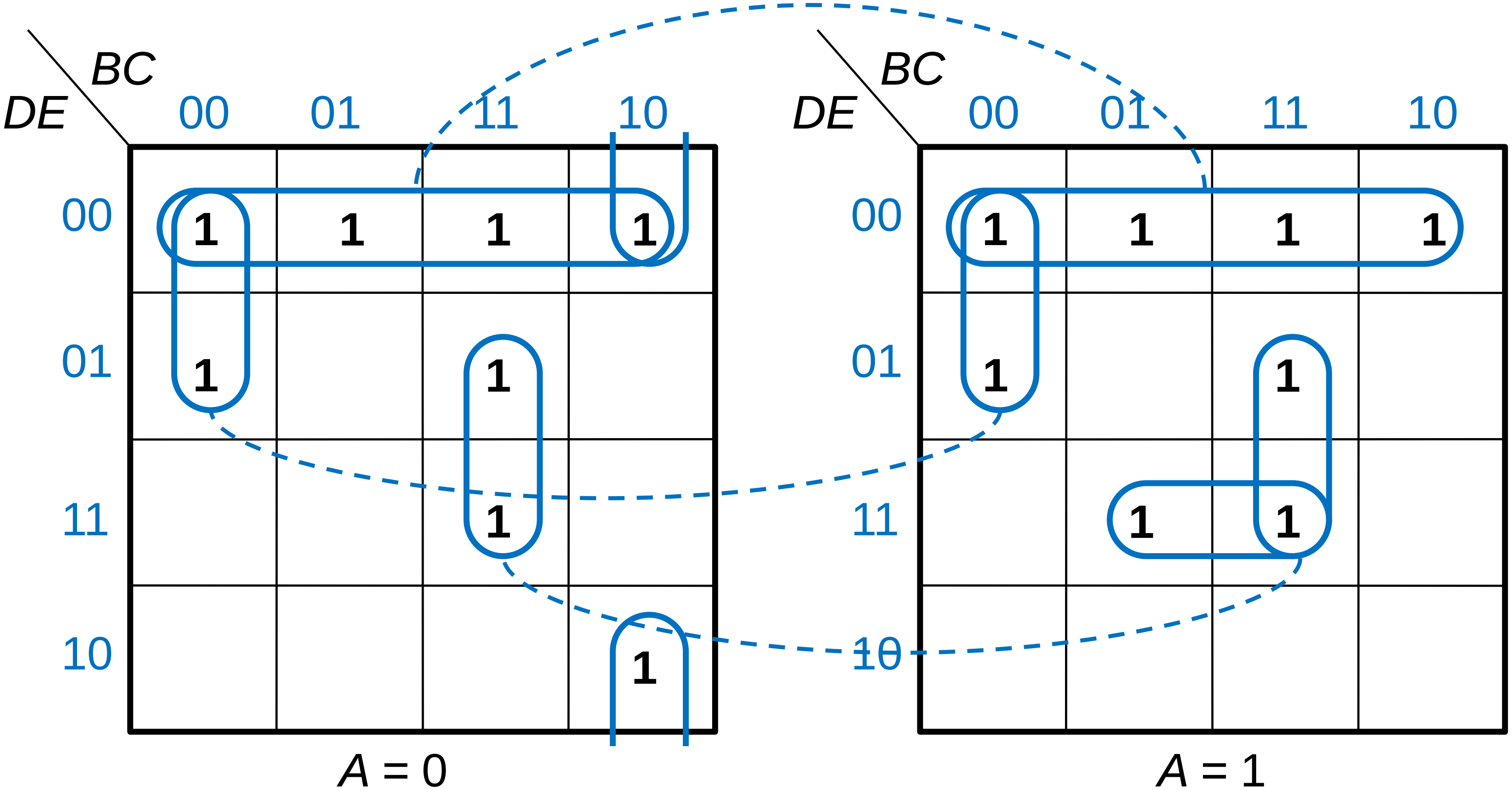
- 4 in the same layer, one in the other layer



Other Forms of 5-Variable K-Maps

Form 1

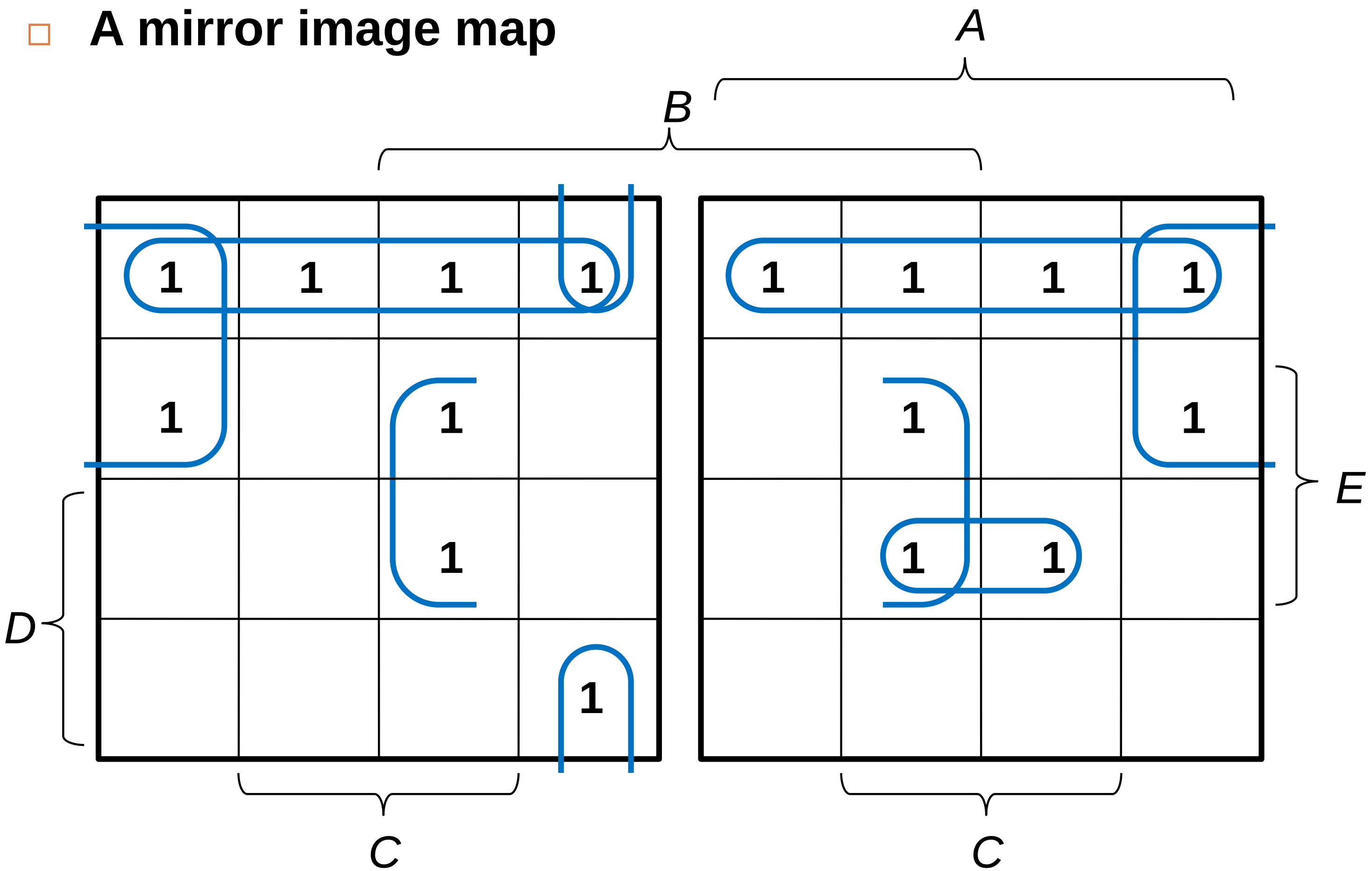
- **Two maps side-by-side**



$F = D'E' + B'C'D' + BCE + A'BC'E' + ACDE$

Form 2

- **A mirror image map**



F = D'E' + B'C'D' + BCE + A'BC'E' + ACDE

Other Uses of Karnaugh Maps

Many operations that can be performed using a truth table or algebraically can be done using a Karnaugh Map (Unit 5.6)