# UNIT 3
## BOOLEAN ALGEBRA (CONT'D)

Spring 2011

# Boolean Algebra (cont'd)

- **Contents**
  - Multiplying out and factoring expressions
  - Exclusive-OR and Exclusive-NOR operations
  - The consensus theorem
  - Summary of algebraic simplification
  - Proving validity of an equation
- **Reading**
  - Unit 3

# Objectives

- **In this unit, you will…**
  - Continue to learn Boolean algebra
    - Simplify, complement, multiply out and factor an expression
    - Prove any theorem in an algebraic way or using a truth table
  - Learn XOR

# Guidelines for Multiplying Out and Factoring

- **Use**

1. $X(Y + Z) = XY + XZ$

2. $(X + Y)(X + Z) = X + YZ$

3. $(X + Y)(X' + Z) = XZ + X'Y$

- **For multiplying out, apply 2. and 3. before 1. to avoid unnecessary terms**

- **For factoring, apply 1., 2., 3. from right terms to left terms**

Boolean Algebra (cont'd)

# Proof?

1. **$(X + Y)(X + Z) = X + YZ$**

   - $(X + Y)(X + Z) = XX + XZ + YX + YZ$
     $$= X + XZ + XY + YZ$$
     $$= X + XY + YZ = X + YZ$$

   - Or,
     $X = 0, (0 + Y)(0 + Z) = YZ$ $\Leftrightarrow$ $0 + YZ = YZ$
     $X = 1, (1 + Y)(1 + Z) = 1 \bullet 1 = 1$ $\Leftrightarrow$ $1 + YZ = 1$

2. **$(X + Y)(X' + Z) = XZ + X'Y$**

   - $(X + Y)(X' + Z) = XX' + XZ + YX' + YZ = 0 + XZ + X'Y + YZ$
     $$= XZ + X'Y + (X' + X)YZ$$
     $$= XZ + X'Y + X'YZ + XYZ$$
     $$= XZ(1 + Y) + X'Y(1 + Z) = XZ + X'Y$$

   expand & simplify

# Multiplying Out

$$
\begin{array}{ll}
1. & X(Y + Z) = XY + XZ \\
2. & (X + Y)(X + Z) = X + YZ \\
3. & (X + Y)(X' + Z) = XZ + X'Y
\end{array}
$$

- **Multiplying out: POS $\Rightarrow$ SOP**
- **Example: $(A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C)$**

  $\qquad$ 2. $\qquad\qquad\qquad$ 3.

  $= (A + B + C'D)(A + B + E)[AC + A'(D' + E)]$

  $\qquad$ 2.

  $= (A + B + C'DE)(AC + A'D' + A'E)$

  $= AC + \cancel{ABC} + A'BD' + A'BE + A'C'DE$

  $= AC + A'BD' + A'BE + A'C'DE$ ∘ ∘ ○

  By brute force
  $\Rightarrow$162 terms!

# Factoring

□ **Factoring: SOP ⇒ POS**

□ **Example: *AC + A'BD' + A'BE + A'C'DE***

$$= \underline{A}\ \underline{C} + \underline{A'}(\underline{BD' + BE + C'DE})$$

$$\text{3.}$$

$$= X\,Z + X'\,Y \qquad\qquad = (X + Y)(X' + Z)$$

$$= (A + BD' + BE + C'DE)(A' + C)$$

$$= [\underline{A + C'DE} + \underline{B}(\underline{D' + E})](A' + C)$$

$$\text{2.}$$

$$= [\qquad X + YZ \qquad](A' + C) = (X + Y)(X + Z)(A' + C)$$

$$= (\underline{A + C'DE + B})(A + \cancel{C'DE} + D' + E)(A' + C)$$

$$\text{2.}$$

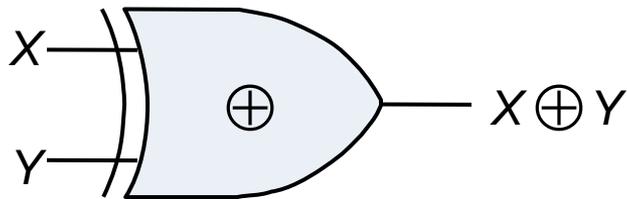$$= (A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C)$$

Boolean Algebra (cont'd)

**8** XOR and XNOR

**Difference and equivalence**

Boolean Algebra (cont'd)

# Operation – Exclusive-OR

- **Exclusive-OR (XOR) ($\oplus$)**
  - $0 \oplus 0 = 0$, $0 \oplus 1 = 1$, $1 \oplus 0 = 1$, $1 \oplus 1 = 0$
  - Symbol

$$X \oplus Y$$

  - Truth table

| $X$ | $Y$ | $X \oplus Y$ |
|-----|-----|--------------|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

check difference of inputs

  - Boolean expression: $X \oplus Y = X'Y + XY'$

show the condition to make output ==1

Boolean Algebra (cont'd)

# Exclusive-OR Operations

- $X \oplus Y = X'Y + XY'$

- **Useful theorems:**
  - $X \oplus 0 = X$
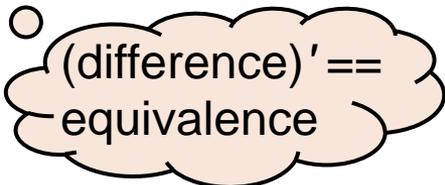  - $X \oplus Y = Y \oplus X$ (commutative)
  - $X \oplus 1 = X'$
  - $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) = X \oplus Y \oplus Z$ (associative)
  - $X \oplus X = 0$
  - $X(Y \oplus Z) = XY \oplus XZ$ (distributive)
  - $X \oplus X' = 1$
  - $(X \oplus Y)' = X \oplus Y' = X' \oplus Y = X'Y' + XY$

  > (difference)' == equivalence

  - Prove distributive law?

    $XY \oplus XZ = XY(XZ)' + (XY)'XZ$ (by definition)

    $\qquad = XY(X' + Z') + (X' + Y')XZ$ (DeMorgan)

    $\qquad = XYZ' + XY'Z$

    $\qquad = X(YZ' + Y'Z)$

    $\qquad = X(Y \oplus Z)$

Boolean Algebra (cont'd)

# Operation – Exclusive-NOR

- **Exclusive-NOR (XNOR) (≡)**
  - $0 \equiv 0 = 1$, $0 \equiv 1 = 0$, $1 \equiv 0 = 0$, $1 \equiv 1 = 1$
  - Symbol

$$X \equiv Y$$

  - Truth table
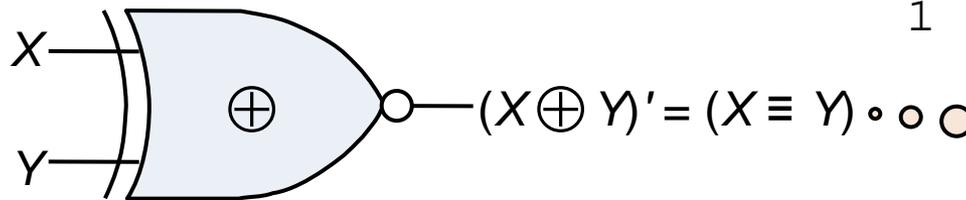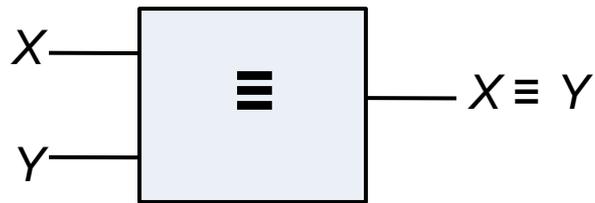
check equivalence of inputs

| $X$ | $Y$ | $X \equiv Y$ | $(X \oplus Y)'$ |
|-----|-----|--------------|-----------------|
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |

$(X \oplus Y)' = (X \equiv Y)$

(difference)′ == equivalence

  - Boolean expression: $X \equiv Y = X'Y' + XY$

show the condition to make output ==1

Boolean Algebra (cont'd)

# Simplification of XOR and XNOR

□ $X \oplus Y = X'Y + XY'$

$X \equiv Y = X'Y' + XY$

$(X'Y + XY')' = X'Y' + XY$

□ e.g., $F = (A'B \equiv C) + (B \oplus AC')$

$= [A'BC + (A'B)'C] + [B'AC' + B(AC')']$

$= A'BC + (A + B')C' + B'AC' + B(A' + C)$

$= B(A'C + A' + C) + C'(A + B' + AB')$

$= B(A' + C) + C'(A + B')$ (can be further simplified!)

□ e.g., $F = A' \oplus B \oplus C$

$= (A'B' + AB) \oplus C$

$= (A'B' + AB)C' + (A'B' + AB)'C$

$= A'B'C' + ABC' + A'BC + AB'C$

Boolean Algebra (cont'd)

# 13 The Consensus Theorem

**Redundancy removal**

# The Consensus Theorem

- $XY + X'Z + YZ = XY + X'Z$ (**YZ is redundant**)
  - Proof: $XY + X'Z + YZ = XY + X'Z + (X + X') YZ$
    $$= (XY + XYZ) + (X'Z + X'YZ)$$
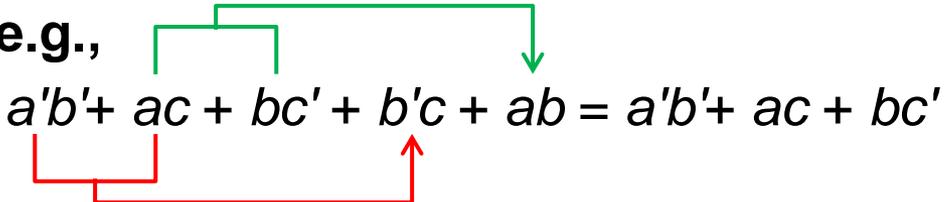    $$= XY(1 + Z) + X'Z(1 + Y) = XY + X'Z$$

- **How to find consensus terms?**
  1. Find a pair of terms, one of which contains a variable and the other contains its complement

     $A'C'D + A'\underline{BD} + BCD + A\underline{BC} + ACD'$ $\qquad$ ($A \leftrightarrow A'$)

  2. Ignore the variable and its complement; the left variables composite the consensus term

     $(A'\underline{BD}) + (A\underline{BC}) \Rightarrow \underline{BD} \bullet \underline{BC} = BCD$ (redundant term)

- **e.g.,**

  $a'b' + ac + bc' + b'c + ab = a'b' + ac + bc'$

# Ordering Does Matter!

- **e.g., *A'C'D + A'BD + BCD + ABC + ACD'***
    - *A'C'D + A'BD + BCD + ABC + ACD'*

    $= A'C'D + A'BD + ABC + ACD'$

    $\Rightarrow$ 4 terms

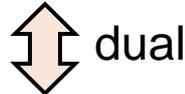    - *A'C'D + A'BD + BCD + ABC + ACD'*

    $= A'C'D + BCD + ACD'$

    $\Rightarrow$ Only 3 terms!

# Dual Form of the Consensus Theorem

- $XY + X'Z + YZ = XY + X'Z \Rightarrow YZ$ **is redundant**

$$\updownarrow \text{ dual}$$

- $(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z) \Rightarrow (Y + Z)$ **is redundant**

- **e.g.,**

  $(a + b + c')(a + b + d')(b + c + d') = (a + b + c')(b + c + d')$

  $(a + b + c') + (b + c + d') \Rightarrow a + b + b + d' = a + b + d'$

Boolean Algebra (cont'd)

# Redundancy Injection

- **e.g., $ABCD + B'CDE + A'B' + BCE'$**
  - Find consensus terms

    $ABCD + B'CDE \Rightarrow ACDE$

    $ABCD + A'B' \Rightarrow \cancel{BCD(B')} + \cancel{(ACD)A'}$

    $B'CDE + BCE' \Rightarrow \cancel{CDE(CE')} + \cancel{B'CD(BC)}$

    $A'B' + BCE' \Rightarrow A'CE'$

  - Add consensus term $ACDE$

    $ABCD + B'CDE + \underline{A}'B' + BC\underline{E}' + \underline{ACDE}$

    $= A'B' + BCE' + ACDE$

Boolean Algebra (cont'd)

# Summary of Algebraic Simplification

**Simplification can reduce cost**

Boolean Algebra (cont'd)

# Rule A -- Combining Terms

- $XY + XY' = X(Y + Y') = X$

- **e.g.,**
  - $abc'd' + abcd' = abd'$ ($X = abd'$, $Y = c$)
  - $ab'c + abc + a'bc = ab'c + abc + abc + a'bc$ (repeat term)

$$= ac + bc$$

  - $(a + bc)(d + e') + a'(b' + c')(d + e') = d + e'$ (why?? DeMorgan)

Boolean Algebra (cont'd)

# Rule B -- Eliminating Terms

1. $X + XY = X$ **(keep the boss)**
2. $XY + X'Z + YZ = XY + X'Z$ **(consensus)**

- **e.g.,**
  - $a'b + a'bc = a'b$ ($X = a'b$)
  - $a'bc' + bcd + a'bd = a'bc' + bcd$ ($X = c$, $Y = bd$, $Z = a'b$)

Boolean Algebra (cont'd)

# Rule C -- Eliminating Literals

- $X + X'Y = (X + X')(X + Y) = X + Y$

- **e.g.,**
  - $A'B + A'B'C'D' + ABCD'$

  $= A'(B + B'C'D') + ABCD'$ (common term: $A'$)

  $= A'(B + C'D') + ABCD'$ (Rule C)

  $= A'B + A'C'D' + ABCD'$

  $= B(A' + ACD') + A'C'D'$ (common term: $B$)

  $= B(A' + CD') + A'C'D'$ (Rule C)
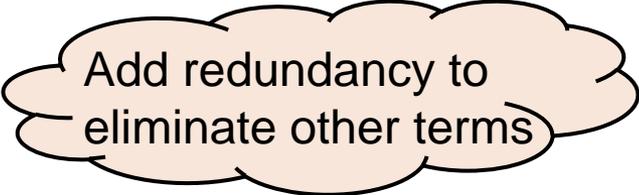
  $= A'B + BCD' + A'C'D'$ (final terms)

Boolean Algebra (cont'd)

# Rule D -- Adding Redundant Terms

1. $Y = Y + XX'$

2. $Y = Y(X + X')$

3. $XY + X'Z = XY + X'Z + YZ$ ∘ ∘ ∘

   Add redundancy to eliminate other terms

4. $X = X + XY$

□ **e.g.,**

   ▫ $WX + XY + X'Z' + WY'Z'$

   $= WX + XY + X'Z' + WY'Z' + WZ'$ (add $WZ'$ by consensus thm)

   $= WX + XY + X'Z' + WZ'$ (eliminate $WY'Z'$ by $WZ'$)

   $= WX + XY + X'Z'$ (consensus again)

Boolean Algebra (cont'd)

# Another Example

- $A'B'C'D' + A'BC'D' + A'BD + A'BC'D + ABCD + ACD' + B'CD'$

  = … (Apply rules A, B, C, D)

  = $A'C'D' + A'BD + B'CD' + ABC$

- **No easy way to determine when a Boolean expression has a minimum # of terms or literals**
  - Systematic way will be discussed in Unit 5 & Unit 6

Boolean Algebra (cont'd)

# Proving Validity of an Equation

Boolean Algebra (cont'd)

# How to Determine if an Equation Valid?

1. **Construct a truth table (proof by cases)**
2. **Manipulate one side until it is identical to the other side**
3. **Reduce both sides independently to the same expression**
4. **Perform the same operation on both sides if the operation is reversible (Boolean algebra ≠ ordinary algebra)**
   - Complement is reversible
   - Multiplication/division and addition/subtraction are not reversible
     - *x*+*y* = *x*+*z* does not imply *y*=*z* (e.g., *x*=1, *y*=0, *z*=1)
     - *xy* = *xz*    does not imply *y*=*z* (e.g., *x*=0, *y*=0, *z*=1)
   - **Using 2. and 3., usually…**
     1. Reduce both sides to (minimum) SOP or POS
     2. Compare both sides
     3. Try to add or delete terms by using theorems

Boolean Algebra (cont'd)

# Example

- **Show that** *A'BD' + BCD + ABC' + AB'D = BC'D' + AD + A'BC*
  - By the consensus theorem,

    *A'BD' + BCD + ABC' + AB'D*

    = *A'BD' + BCD + ABC' + AB'D + A'BC + BC'D' + ABD*

    = *AD + A'BD' + BCD + ABC' + A'BC + BC'D'*

    = *AD + A'BC + BC'D'*

Boolean Algebra (cont'd)

# One More Example

- **Show that** $A'BC'D + (A' + BC)(A + C'D') + BC'D + A'BC'$

    $= ABCD + A'C'D' + ABD + ABCD' + BC'D$

1. Reduce the left side:

    $A'BC'D + (A' + BC)(A + C'D') + BC'D + A'BC'$

    $= (A' + BC)(A + C'D') + BC'D + A'BC'$

    $= A'C'D' + ABC + BC'D + A'BC'$ (multiplying out)

    $= A'C'D' + ABC + BC'D$ (consensus)

2. Reduce the right side:

    $ABCD + A'C'D' + ABD + ABCD' + BC'D$

    $= ABC + A'C'D' + ABD + BC'D$

    $= ABC + A'C'D' + BC'D$ (consensus)

3. Because both sides were independently reduced to the same expression, the original equation is valid.