UNIT 2
BOOLEAN ALGEBRA

Spring 2011

Boolean Algebra

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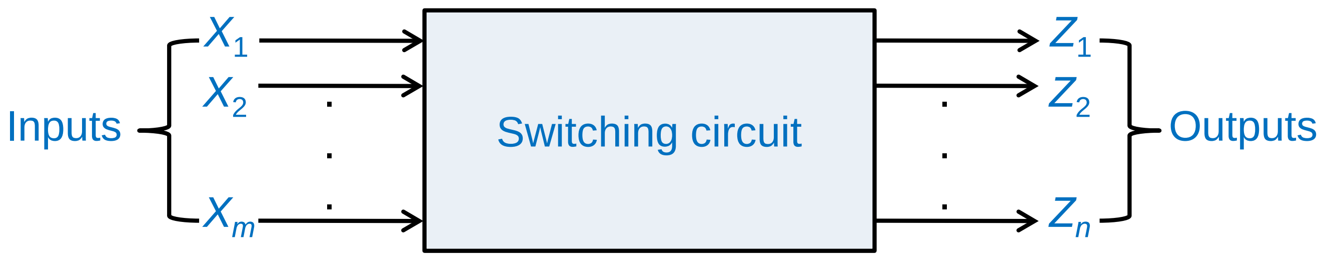
  - Unit 2

Introduction

- **Boolean algebra**
  - Is the basic mathematics for logic design of digital systems
  - Differs from ordinary algebra in the values, operations, and laws
- **History**
  - George Boole developed Boolean algebra in 1847 and used it to solve problems in mathematical logic
    - British mathematician and philosopher
  - Claude Shannon first applied Boolean algebra to the design of switching circuits in 1939
    - American electrical engineer and mathematician
    - Master's thesis (21 years-old)
- **In this unit, you will learn how to…**
  - Use a truth table
  - Manipulate basic operations and apply laws of Boolean algebra
  - Relate Boolean expressions to basic logic gates

Switching Circuit & Boolean Variables

- **A switching circuit has one or more inputs and one or more outputs that take on discrete values (two-value in general)**



- **We usually use a Boolean variable, such as *X*, *Y*, *Z*, to represent an input or output of a switching circuit**
  - Usually take on only two different values
  - 1/0 for High/Low or True/False or Yes/No
    - Just symbols, NO numeric values
  - A two-value Boolean variable is also called a switching variable

Boolean algebra differs from ordinary algebra in values, operations, laws

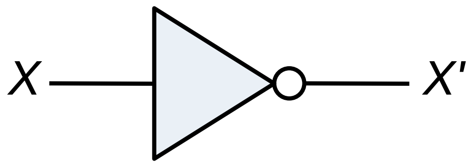
Boolean algebra differs from ordinary algebra in values, operations, laws

Basic Operations

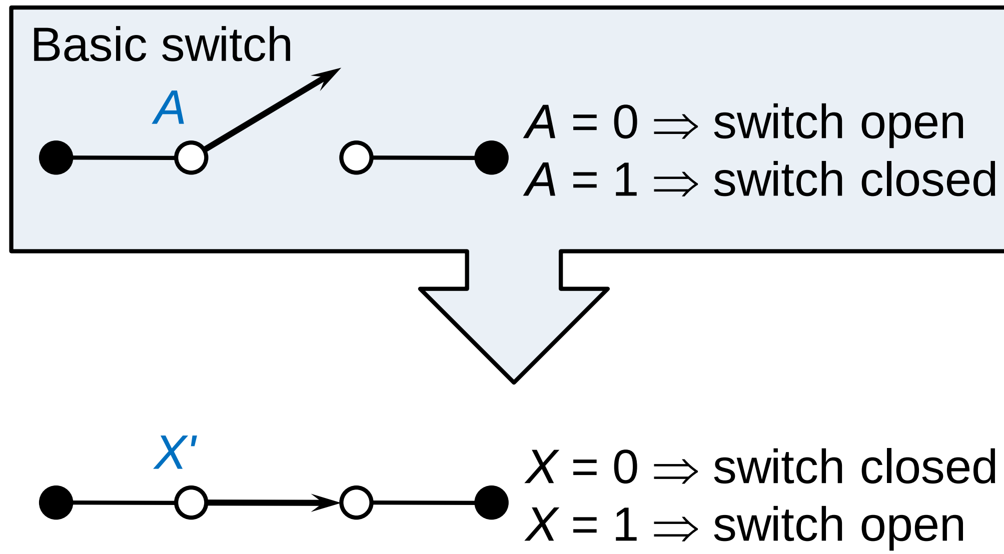
NOT, AND, OR

Operation -- Logic NOT

- **Complement = Inverse = Negate = NOT (' ; ¯; ~; ¬)**
  - 0'=1, 1'=0
  - Symbol



NOT gate
Inverter



  - Truth table

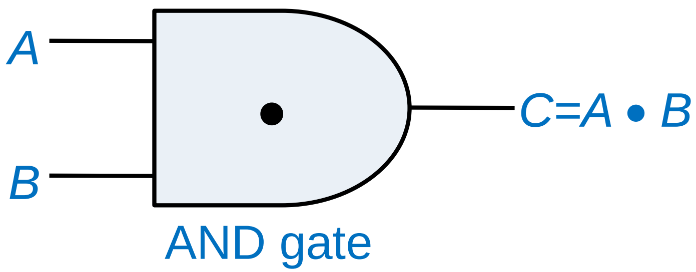
Inputs → X; Outputs → X'

X	X'
0	1
1	0

Input combinations; Output values

Operation -- Logic AND

- **AND (•; ∧)**
  - $0 \bullet 0 = 0,\ 0 \bullet 1 = 0,\ 1 \bullet 0 = 0,\ 1 \bullet 1 = 1$
  - Symbol



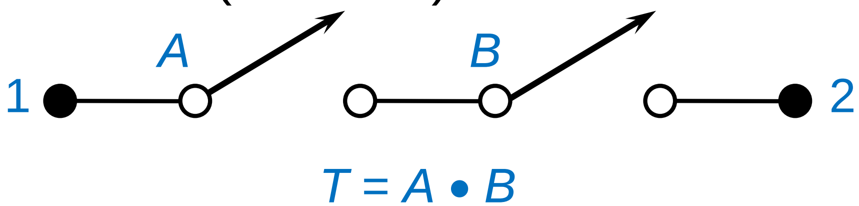
  - Truth table

Omit "•"

A	*B*	*C* = *A* *B*
0	0	0
0	1	0
1	0	0
1	1	1

If one input is 0 ⇒ output is 0

  - Switch (in series)

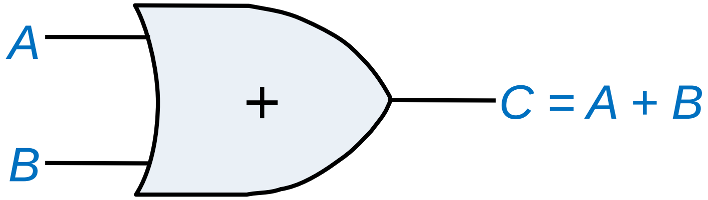


$T = 0 \Rightarrow 1 \rightarrow 2$ open
$T = 1 \Rightarrow 1 \rightarrow 2$ closed

Switch *A* closed and switch *B* closed

Operation -- Logic OR

- **OR (+; ∨)**
  - 0 + 0 = 0, 0 + 1 = 1, 1 + 0 = 1, 1 + 1 = 1
  - Symbol



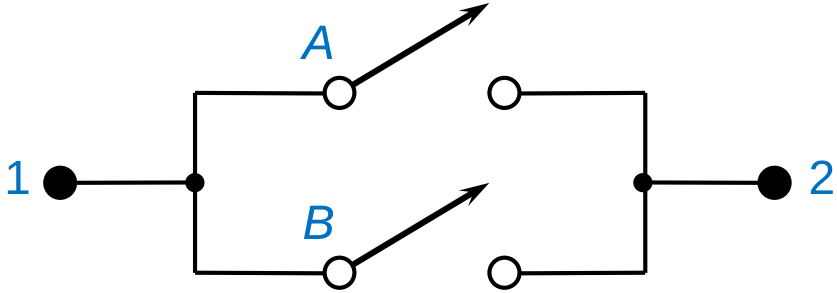
OR gate

  - Truth table

A	B	C = A+B
0	0	0
0	1	1
1	0	1
1	1	1

If one input is 1 ⇒ output is 1

  - Switch (in parallel)

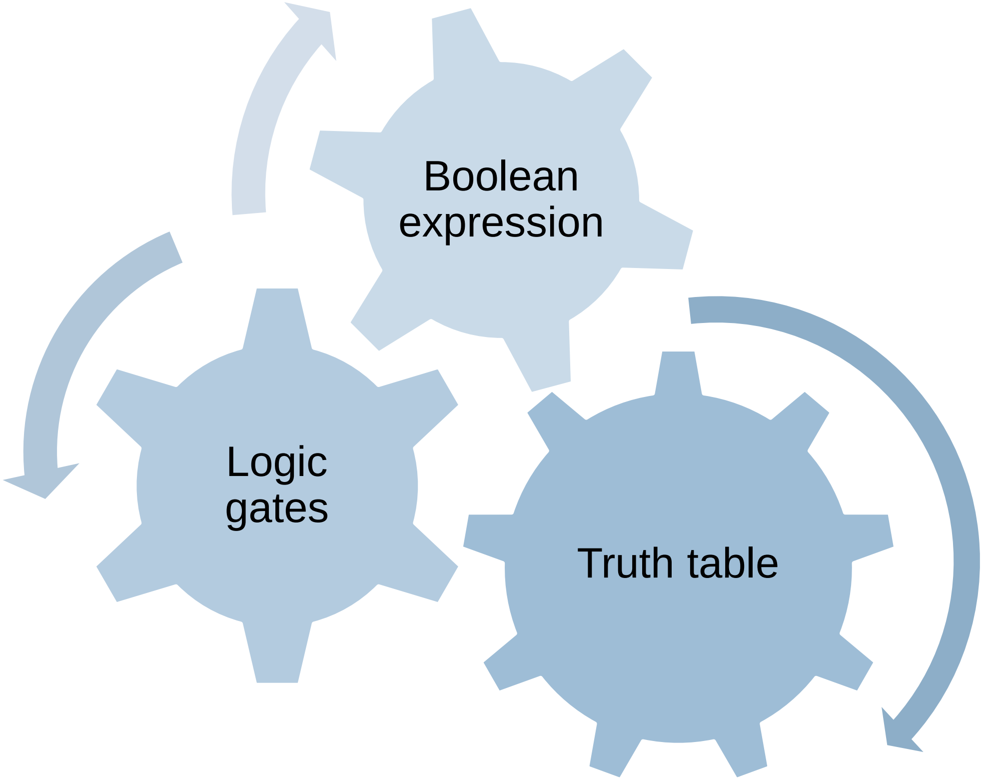


$T = A + B$

$T = 0 \Rightarrow 1 \rightarrow 2$ open
$T = 1 \Rightarrow 1 \rightarrow 2$ closed

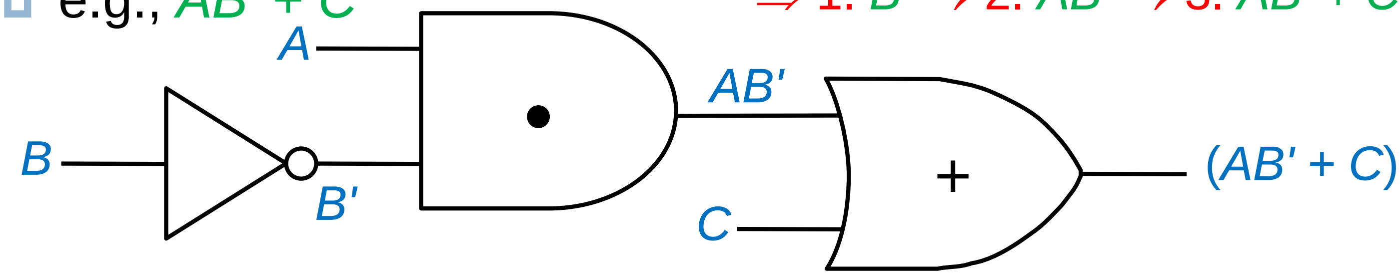
Switch *A* closed or switch *B* closed

Boolean Expressions and Truth Tables

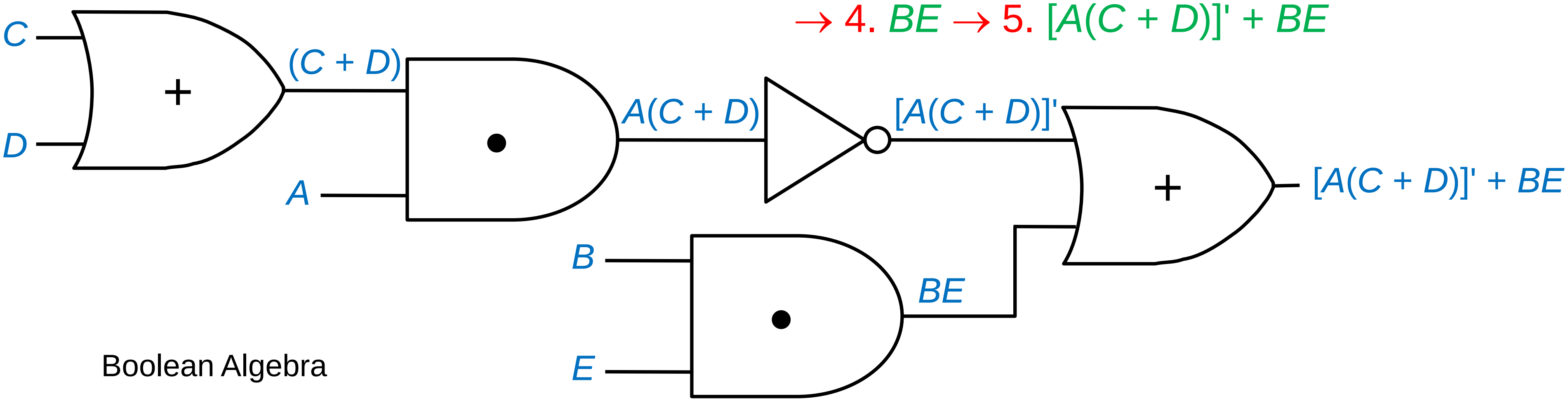


Boolean Expressions vs. Logic Gates

- A **Boolean expression** is formed by basic operations on variables or constants, e.g., the simplest one: 0, 1, *X*, *Y*'
- **Realize a Boolean expression by a circuit of logic gates**
  - Perform operations in order: Parentheses → NOT → AND → OR
  - e.g., *AB*' + *C* ⇒ 1. *B*' → 2. *AB*' → 3. *AB*' + *C*



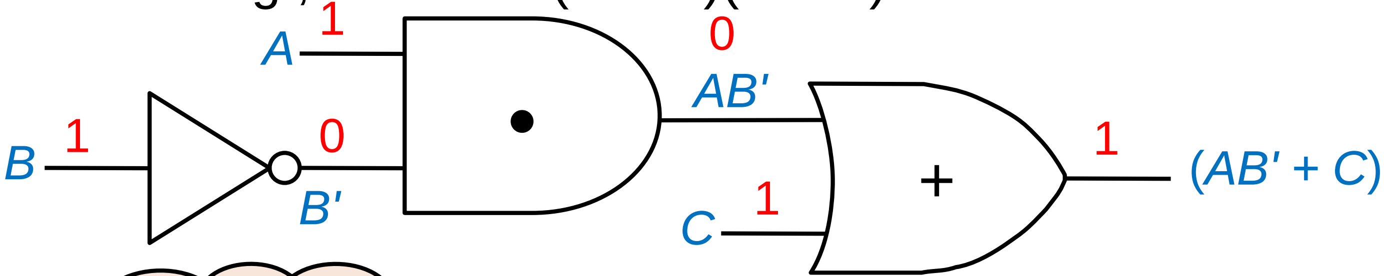
  - e.g., [*A*(*C* + *D*)]' + *BE* ⇒ 1. *C* + *D* → 2. *A*(*C* + *D*) → 3. [*A*(*C* + *D*)]' → 4. *BE* → 5. [*A*(*C* + *D*)]' + *BE*



Boolean Expressions vs. Truth Tables

n variables $\Rightarrow 2^n$ rows

- A **truth table** specifies the output values of a Boolean expression for **all possible combinations of input values**
- $\Rightarrow$ **Check the equivalence between two expressions**
  - e.g., $AB' + C = (A + C)(B' + C)$

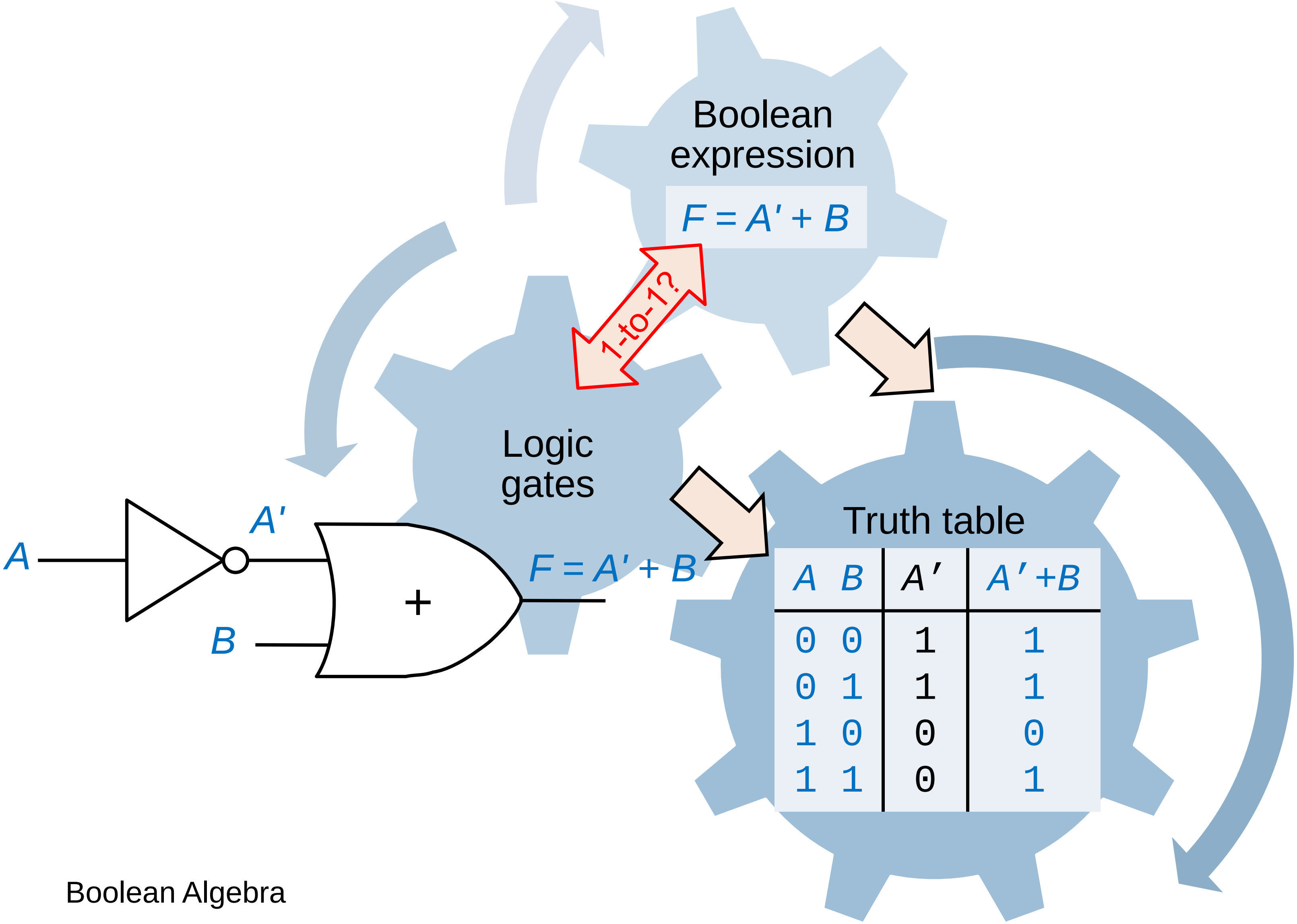


One function has different expressions

Expressions show the condition to make output ==1

A	B	C	B′	AB′	AB′ + C	A+C	B′+C	(A+C)(B′+C)
0	0	0	1	0	0	0	1	0
0	0	1	1	0	1	1	1	1
0	1	0	0	0	0	0	0	0
0	1	1	0	0	1	1	1	1
1	0	0	1	1	1	1	1	1
1	0	1	1	1	1	1	1	1
1	1	0	0	0	0	1	0	0
1	**1**	**1**	**0**	**0**	**1**	1	1	1

Example: $F = A' + B$



A	B	A'	A' + B
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

Boolean algebra differs from ordinary algebra in values, operations, laws

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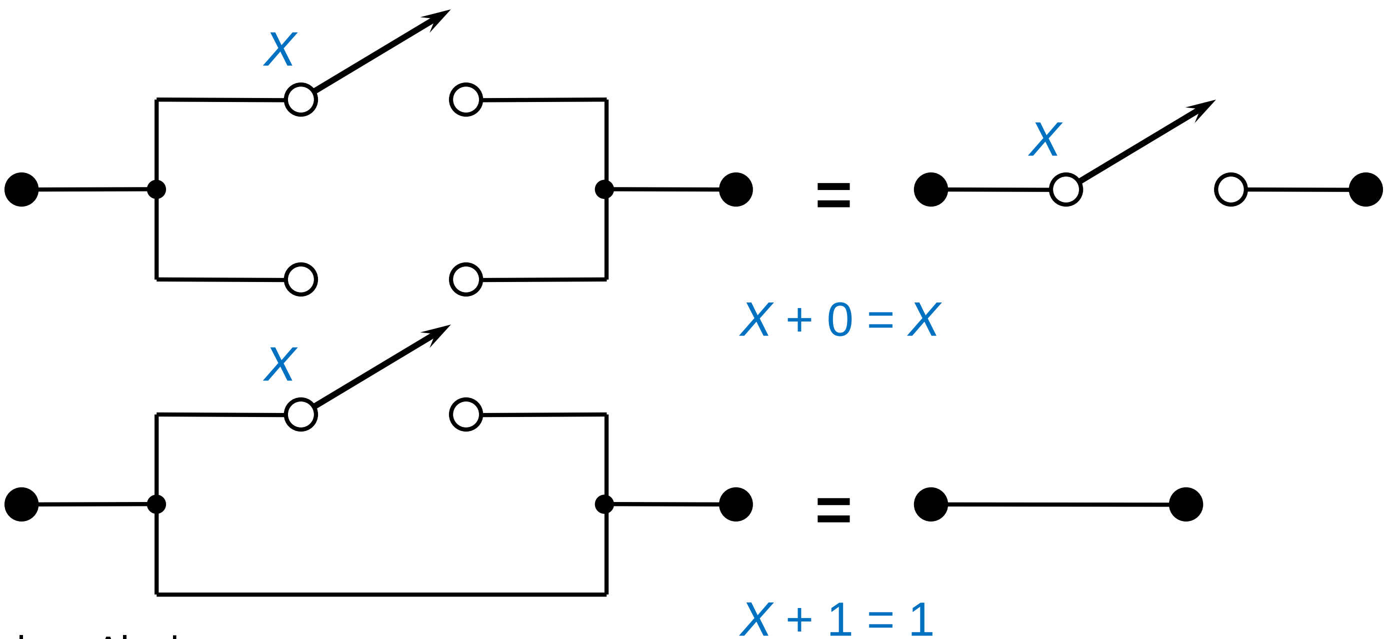
Theorems and Laws

Basic Theorems (1/3)

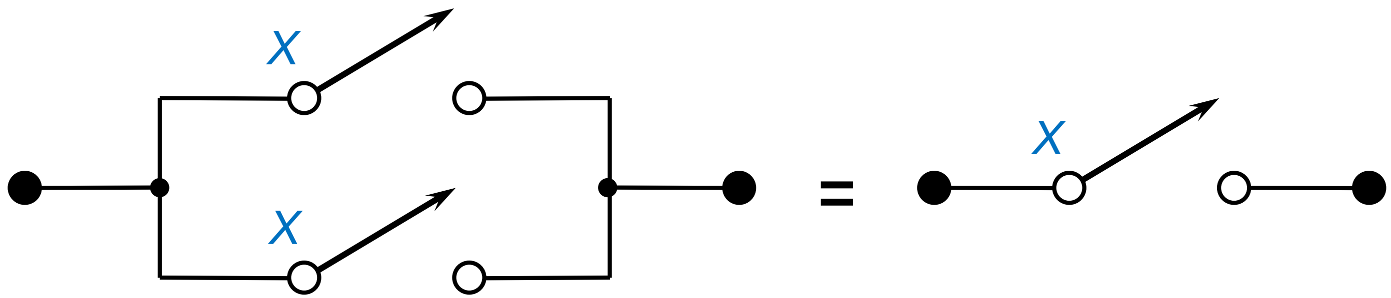
- **Operations with 0 and 1**
  - $X + 0 = X$
  - $X \bullet 1 = X$
  - $X + 1 = 1$
  - $X \bullet 0 = 0$
  - e.g., $(AB' + D)E + 1 = 1$

The variable *X* can be substituted by any expression

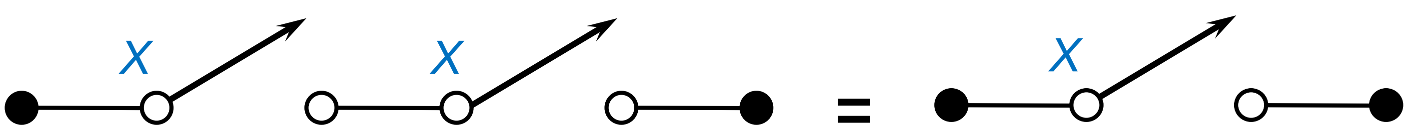


Basic Theorems (2/3)

- **Idempotent laws**
  - $X + X = X$
  - $X \bullet X = X$



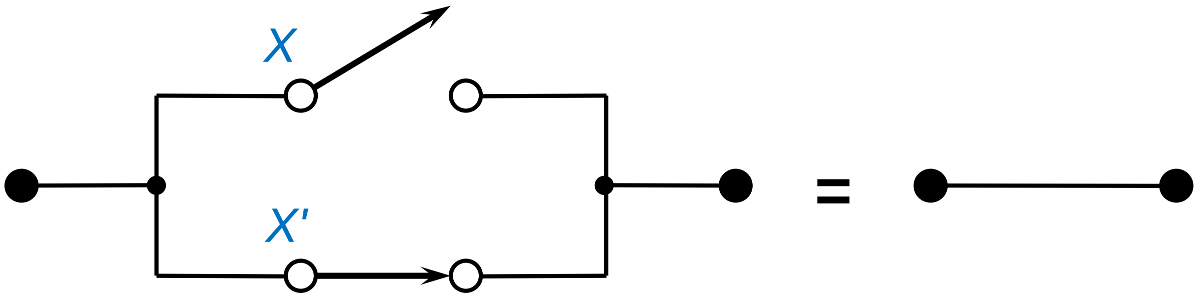
$X + X = X$



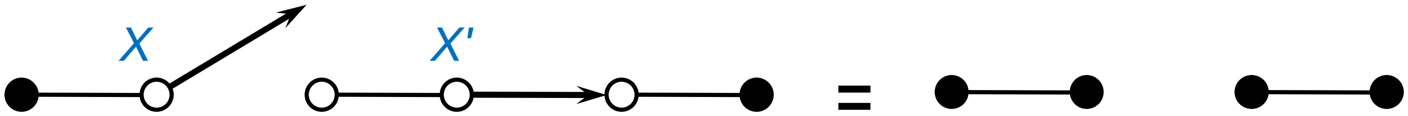
$X \bullet X = X$

Basic Theorems (3/3)

- **Involution law**
  - $(X')' = X$
- **Laws of complementarity**
  - $X + X' = 1$
  - $X \bullet X' = 0$
  - e.g., $(AB' + D)(AB' + D)' = 0$



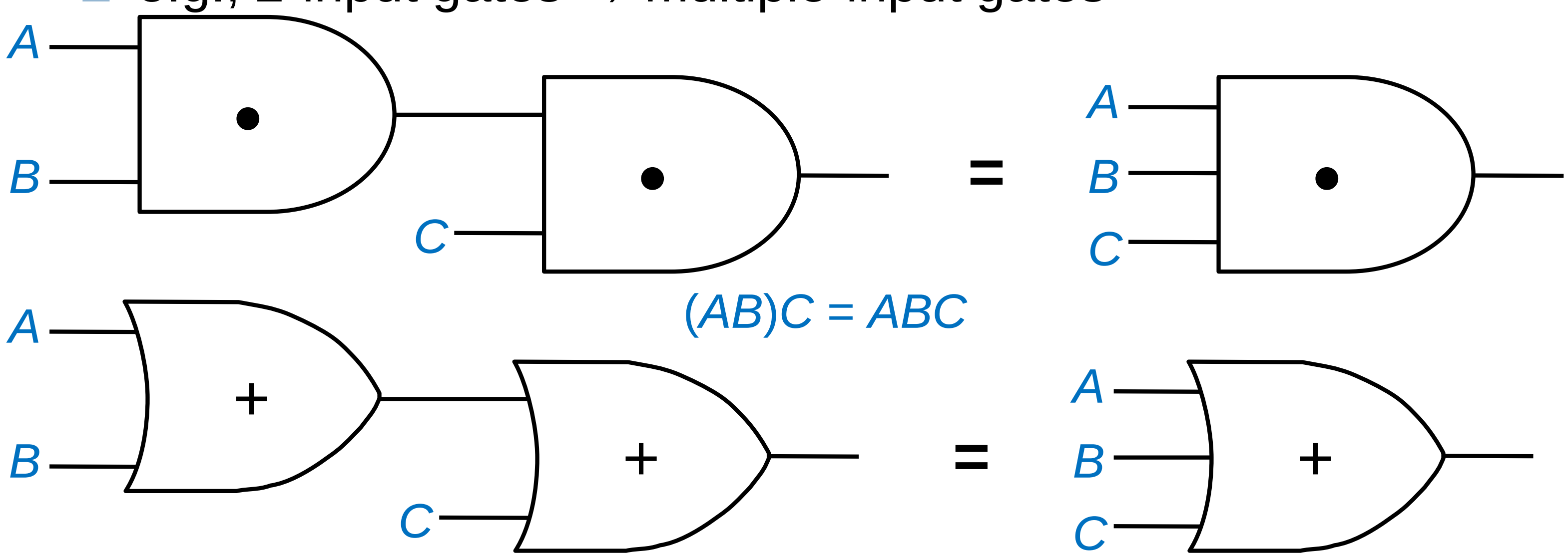
$X + X' = 1$



$X \bullet X' = 0$

Commutative/Associative Laws

- **Commutative laws for AND and OR**
  - $XY = YX$
  - $X + Y = Y + X$
- **Associative laws for AND and OR**
  - $(XY)Z = X(YZ) = XYZ$
  - $(X + Y) + Z = X + (Y + Z) = X + Y + Z$
  - e.g., 2-input gates ⇒ multiple-input gates



$(AB)C = ABC$

$(A + B) + C = A + B + C$

Distributive Laws (1/2)

- **Ordinary distributive law**
  - $X(Y + Z) = XY + XZ$
- **Second distributive law (Important !)**
  - $X + YZ = (X + Y)(X + Z)$
  - Proof?

Only valid for Boolean algebra

Distributive Laws (2/2)

- **Prove a Boolean theorem/law by:**
  1. Truth table

     $X + YZ = (X + Y)(X + Z)$

     =?

X	Y	Z	YZ	X+YZ	X+Y	X+Z	(X+Y)(X+Z)
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

  2. Basic theorems

$(X + Y)(X + Z)$

$= X(X + Z) + Y(X + Z)$

$= XX + XZ + YX + YZ$

$= X + XZ + XY + YZ$

$= X\bullet 1 + XZ + XY + YZ$

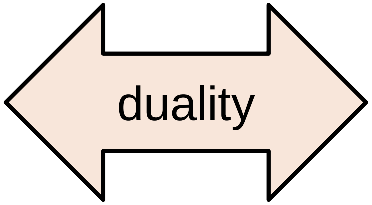
$= X(1+Z+Y) + YZ$

$= X\bullet 1 + YZ$

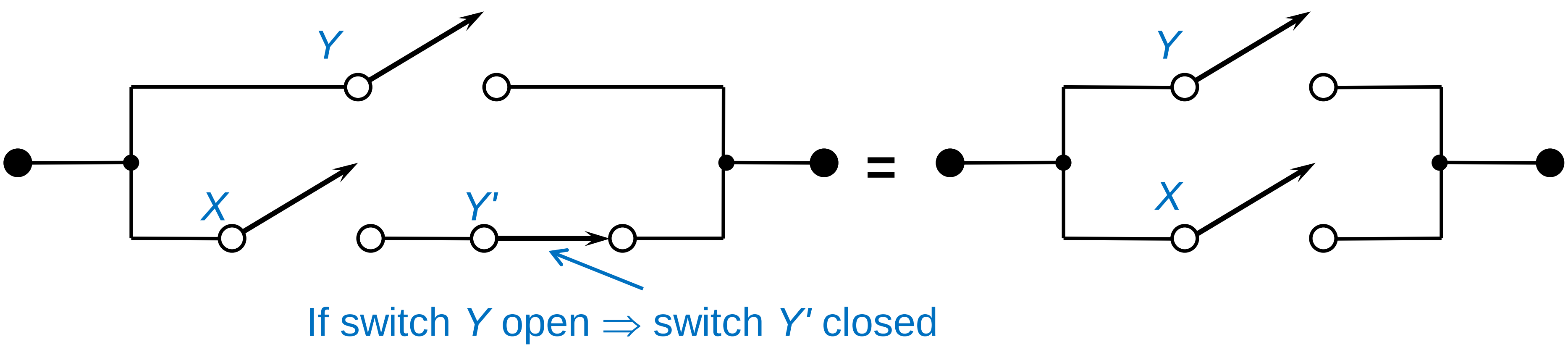
$= X + YZ$

Simplification Theorems

- **Useful simplification theorems**
  - $XY + XY' = X$
  - $X + XY = X$
  - $(X+Y')\,Y = XY$

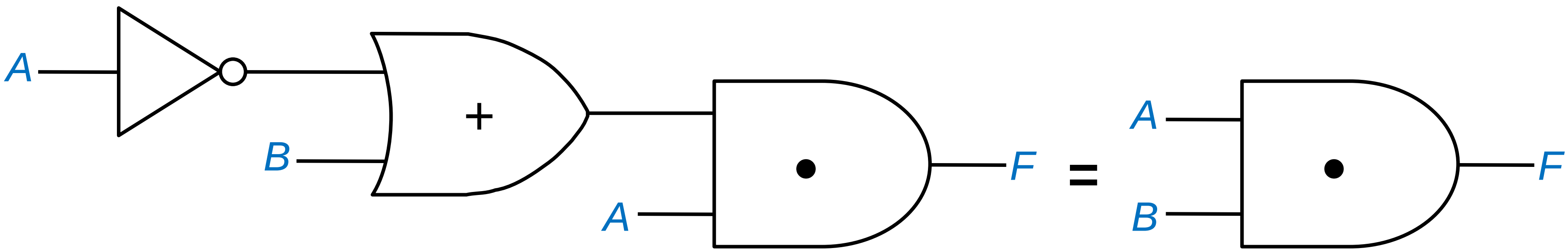


  - $(X+Y)(X+Y') = X$
  - $X(X+Y) = X$
  - $XY' + Y = X + Y$
- **Proof:**
  - $X + XY = X\bullet 1 + XY = X(1+Y) = X\bullet 1 = X$
  - $X(X+Y) = XX + XY = X + XY = X$
  - $XY' + Y = Y + XY' = (Y+X)(Y+Y') = (Y+X)\bullet 1 = Y + X$
    - Use switches



Simplification Examples

1. $A(A'+B) = AB$



2. $Z = [A + B'C + D + EF][A + B'C + (D + EF)']$

$= [\underline{A + B'C} + \underline{D + EF}][\underline{A + B'C} + (\underline{D + EF})']$

$= [\quad X \quad + \quad Y \quad][\quad X \quad + \quad Y' \quad]$

$= X = A + B'C$

3. $Z = (AB + C)(B'D + C'E') + (AB + C)'$

$= (\underline{AB + C})(B'D + C'E') + (\underline{AB + C})'$

$= \quad X \quad Y \quad + \quad X' \quad = XY + X' \bullet 1 = XY + X'(1+Y)$

$= XY + X' + X'Y = (X+X')Y + X'$

$= Y + X' = B'D + C'E' + (AB + C)'$

All roads lead to Rome

Multiplying Out

SOP: Sum of products of only single variables

- **Use the ordinary distributive law**

  $$X(Y + Z) = XY + XZ$$

  to multiply out an expression to obtain a sum-of-products form

  - e.g.,

    $AB' + CD'E + AC'E'$ (V)

    $A + B' + C + D'E$ (V)

    $(A + B)CD + EF$ (X)

Example: Multiplying Out $(A+BC)(A+D+E)$

1. **Multiply out completely and then eliminate redundant terms**

$$(A+BC)(A+D+E) = A+AD+AE+ABC+BCD+BCE$$
$$= A(1+D+E+BC)+BCD+BCE$$
$$= A+BCD+BCE$$

2. **Or, apply 2nd distributive law first: $(X+Y)(X+Z)=X+YZ$**

$$(A+BC)(A+D+E) = A+BC(D+E)$$
$$= A+BCD+BCE$$

2nd distributed law saves your time!

Factoring

POS: Products of sums of only single variables

- **Use the second distributive law**

  $$X + YZ = (X + Y)(X + Z)$$

  to factor an expression to obtain a product-of-sums form

  - e.g.,

    $(A + B')(C + D + E)(A + C' + E')$ (V)

    $(A + B)(C + D + E)F$ (V)

    $AB'C(D' + E)$ (V)

    $(A + B)(C + D) + EF$ (X)

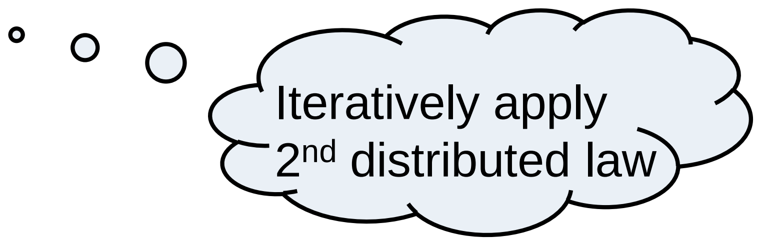
Example: Factoring

1. **Factor $A + B'CD$**

$$A + B'CD = (A + B')(A + CD) = (A + B')(A + C)(A + D)$$

2. **Factor $AB' + C'D$**

$$AB' + C'D = (AB' + C')(AB' + D) = (A + C')(B' + C')(A + D)(B' + D)$$

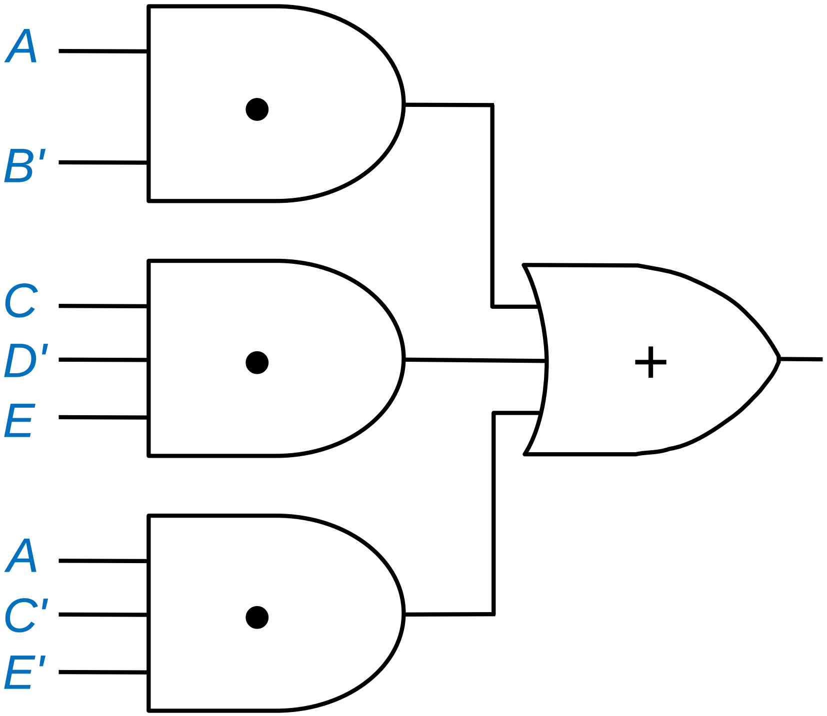


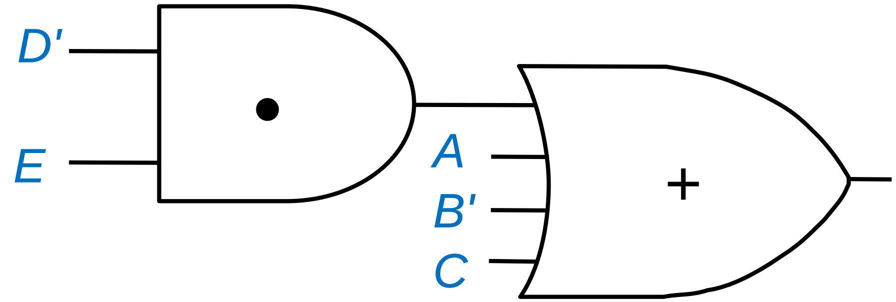
3. **DIY: Factor $C'D + C'E' + G'H$**

SOP vs. Logic Gates

- **Realize SOPs by two-level circuits (AND-OR)**
  - $AB' + CD'E + AC'E'$
  - $A + B' + C + D'E$

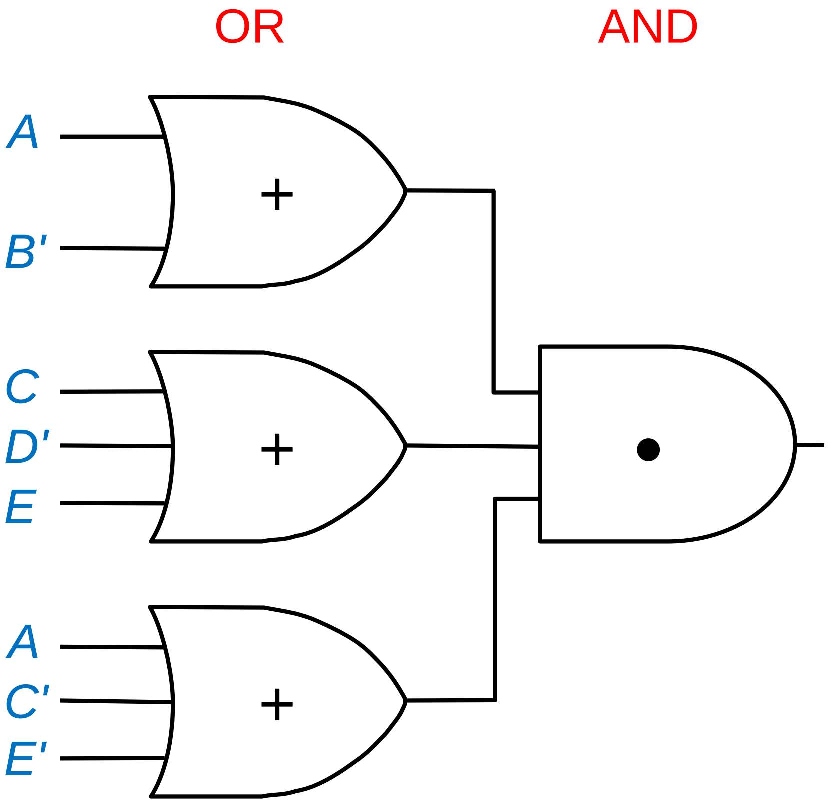
AND OR AND OR

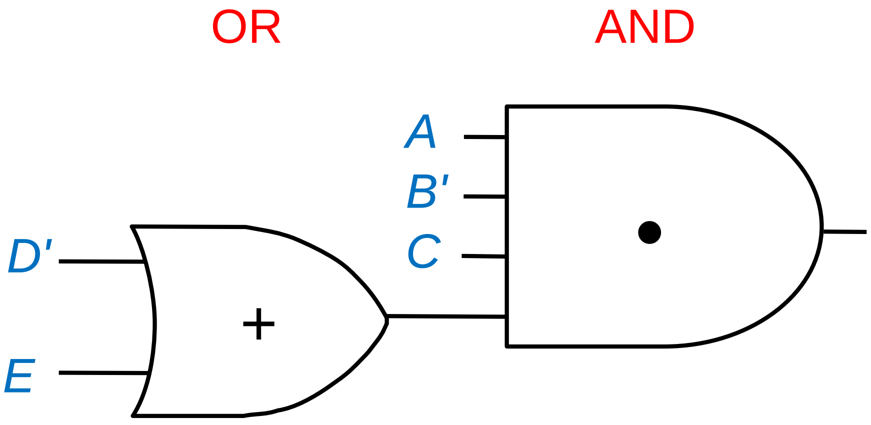




POS vs. Logic Gates

- **Realize POSs by two-level circuits (OR-AND)**
  - $(A + B')(C + D + E)(A + C' + E')$
  - $AB'C(D' + E)$





DeMorgan's Laws

- **Complement a Boolean expression by DeMorgan's laws**
  - $(X + Y)' = X'Y'$
  - $(XY)' = X' + Y'$
  - Proof: By truth table

X Y	X'	Y'	X+Y	(X+Y)'	X'Y'	XY	(XY)'	X'+Y'
0 0	1	1	0	1	1	0	1	1
0 1	1	0	1	0	0	0	1	1
1 0	0	1	1	0	0	0	1	1
1 1	0	0	1	0	0	1	0	0

- **Generalize to *n* variables:**
  - $(X_1+X_2+\ldots+X_n)' = X_1'X_2'\ldots X_n'$
  - $(X_1X_2\ldots X_n)' = X_1'+X_2'+\ldots+X_n'$
- **One-step rule:**
  - $[f(x_1,x_2,\ldots,x_n,0,1,+,\bullet)]' = f(x_1', x_2',\ldots,x_n',1,0,\bullet,+)$
    - $x \leftrightarrow x'$; $+ \leftrightarrow \bullet$; $0 \leftrightarrow 1$

Example: Complementing $(AB' + C)D' + E$

1. **Iteratively apply DeMorgan's laws:**

$$[(AB' + C)D' + E]' = [(AB' + C)D']'E'$$
$$= [(AB' + C)' + D]E'$$
$$= [(AB')'C' + D]E'$$
$$= [(A' + B)C' + D]E'$$

> NOT is applied only to single variables

2. **Or, use one-step rule:**

$$[(AB' + C)D' + E]' = [(((A \bullet B') + C) \bullet D') + E]'$$
$$= (((A' + B) \bullet C') + D) \bullet E'$$

Duality

- **Dual:**
  - $[f(x_1,x_2,\ldots,x_n,0,1,+,\bullet)]^D = f(x_1,x_2,\ldots,x_n,1,0,\bullet,+)$
    - $+ \leftrightarrow \bullet$; $0 \leftrightarrow 1$
- **Cf. DeMorgan's laws:**
  - $[f(x_1,x_2,\ldots,x_n,0,1,+,\bullet)]' = f(x_1',x_2',\ldots,x_n',1,0,\bullet,+)$
    - $x \leftrightarrow x'$; $+ \leftrightarrow \bullet$; $0 \leftrightarrow 1$
- **$\Rightarrow$ Find the dual of an expression:**
  - Complement the entire expression
  - Complement each individual variable
- **e.g., $(XYZ\ldots)^D = X + Y + Z + \ldots$**
- **e.g., $(AB' + C)^D = ?$**

  $(AB' + C)' = (A'+B)C' \Rightarrow (AB' + C)^D = (A+B')C$
- **Application: $F = G \Leftrightarrow F^D = G^D$**