UNIT 3
BOOLEAN ALGEBRA (CONT'D)

Fall 2021

Boolean Algebra (cont'd)

- **Contents**
  - Multiplying out and factoring expressions
  - Exclusive-OR and Exclusive-NOR operations
  - The consensus theorem
  - Summary of algebraic simplification
  - Proving validity of an equation
- **Reading**
  - Unit 3

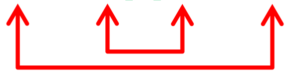
Objectives

- **In this unit, you will…**
  - Continue to learn Boolean algebra
    - Simplify, complement, multiply out and factor an expression
    - Prove any theorem in an algebraic way or using a truth table
  - Learn XOR

Guidelines for Multiplying Out and Factoring

- **Use**

1. $X(Y + Z) = XY + XZ$
2. $(X + Y)(X + Z) = X + YZ$
3. $(X + Y)(X' + Z) = XZ + X'Y$



- **For multiplying out, apply 2. and 3. before 1. to avoid unnecessary terms**
- **For factoring, apply 1., 2., 3. from right terms to left terms**

Proof?

1. **$(X + Y)(X + Z) = X + YZ$**
   - $(X + Y)(X + Z) = XX + XZ + YX + YZ$
     $= X + XZ + XY + YZ$
     $= X + XY + YZ = X + YZ$
   - Or,
     $X = 0$, $(0 + Y)(0 + Z) = YZ \Leftrightarrow 0 + YZ = YZ$
     $X = 1$, $(1 + Y)(1 + Z) = 1 \bullet 1 = 1 \Leftrightarrow 1 + YZ = 1$
2. **$(X + Y)(X' + Z) = XZ + X'Y$**
   - $(X + Y)(X' + Z) = \underline{XX'} + XZ + \underline{YX'} + YZ = 0 + XZ + X'Y + YZ$
     $= XZ + X'Y + (X' + X)YZ$ (expand & simplify)
     $= XZ + X'Y + X'YZ + XYZ$
     $= XZ(1 + Y) + X'Y(1 + Z) = XZ + X'Y$

Multiplying Out

1. $X(Y + Z) = XY + XZ$
2. $(X + Y)(X + Z) = X + YZ$
3. $(X + Y)(X' + Z) = XZ + X'Y$

- **Multiplying out: POS ⇒ SOP**
- **Example:** $(A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C)$

2. 3.

$$= (A + B + C'D)(A + B + E)[AC + A'(D' + E)]$$

2.

$$= (A + B + C'DE)(AC + A'D' + A'E)$$

$$= AC + \cancel{ABC} + A'BD' + A'BE + A'C'DE$$

$$= AC + A'BD' + A'BE + A'C'DE$$

By brute force
⇒162 terms!

Factoring

1. $X(Y + Z) = XY + XZ$
2. $(X + Y)(X + Z) = X + YZ$
3. $(X + Y)(X' + Z) = XZ + X'Y$

- **Factoring: SOP ⇒ POS**
- **Example: $AC + A'BD' + A'BE + A'C'DE$**

$= A\,C + A'(BD' + BE + C'DE)$

3.

$= X\,Z + X'\,Y \qquad = (X + Y)(X' + Z)$

$= (A + BD' + BE + C'DE)(A' + C)$

$= [A + C'DE + B(D' + E)](A' + C)$

2.

$= [\quad X + Y\,Z \quad](A' + C) = (X + Y)(X + Z)(A' + C)$

$= (A + C'DE + B)(A + \sout{C'DE} + D' + E)(A' + C)$

2.

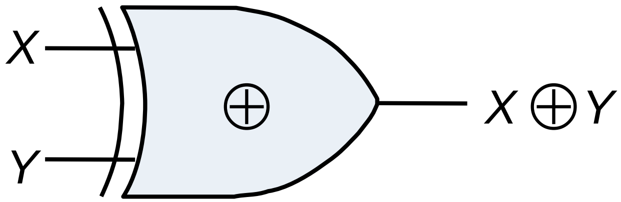
$= (A + B + C')(A + B + D)(A + B + E)(A + D' + E)(A' + C)$

XOR and XNOR

Difference and equivalence

Operation – Exclusive-OR

- **Exclusive-OR (XOR) (⊕)**
  - 0 ⊕ 0 = 0, 0 ⊕ 1 = 1, 1 ⊕ 0 = 1, 1 ⊕ 1 = 0
  - Symbol



  - Truth table

X	Y	X ⊕ Y
0	0	0
0	1	1
1	0	1
1	1	0

check difference of inputs

  - Boolean expression: $X \oplus Y = X'Y + XY'$

show the condition to make output ==1

Exclusive-OR Operations

- $\boldsymbol{X \oplus Y = X'Y + XY'}$
- **Useful theorems:**
  - $X \oplus 0 = X$
  - $X \oplus 1 = X'$
  - $X \oplus X = 0$
  - $X \oplus X' = 1$
  - $X \oplus Y = Y \oplus X$ (commutative)
  - $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) = X \oplus Y \oplus Z$ (associative)
  - $X(Y \oplus Z) = XY \oplus XZ$ (distributive)
  - $(X \oplus Y)' = X \oplus Y' = X' \oplus Y = X'Y' + XY$

(difference)$'$ == equivalence

  - Prove distributive law?

$XY \oplus XZ = XY(XZ)' + (XY)'XZ$ (by definition)

$= XY(X' + Z') + (X' + Y')XZ$ (DeMorgan)

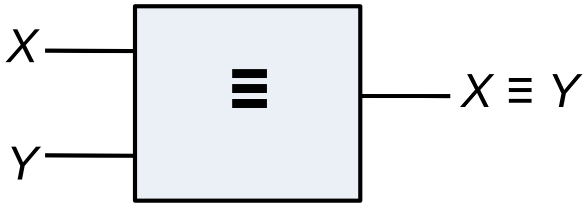
$= XYZ' + XY'Z$

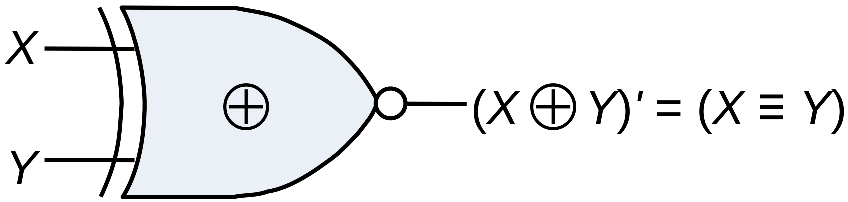
$= X(YZ' + Y'Z)$

$= X(Y \oplus Z)$

Operation – Exclusive-NOR

- **Exclusive-NOR (XNOR) (≡)**
  - $0 \equiv 0 = 1, 0 \equiv 1 = 0, 1 \equiv 0 = 0, 1 \equiv 1 = 1$
  - Symbol





(difference)′ == equivalence

  - Truth table

check equivalence of inputs

X	Y	$X \equiv Y$	$(X \oplus Y)'$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	1	1

  - Boolean expression: $X \equiv Y = X'Y' + XY$

show the condition to make output ==1

Simplification of XOR and XNOR

- $X \oplus Y = X'Y + XY'$
  $X \equiv Y = X'Y' + XY$
  $(X'Y + XY')' = X'Y' + XY$
- e.g., $F = (A'B \equiv C) + (B \oplus AC')$
  $= [A'BC + (A'B)'C'] + [B'AC' + B(AC')']$
  $= A'BC + (A + B')C' + B'AC' + B(A' + C)$
  $= A'BC + AC' + B'C' + B'AC' + A'B + BC$
  $= B(A'C + A' + C) + C'(A + B' + AB')$
  $= B(A' + C) + C'(A + B')$ (can be further simplified!)
- e.g., $F = A' \oplus B \oplus C$
  $= (A'B' + AB) \oplus C$
  $= (A'B' + AB)C' + (A'B' + AB)'C$
  $= A'B'C' + ABC' + A'BC + AB'C$

The Consensus Theorem

Redundancy removal

The Consensus Theorem

- **$XY + X'Z + YZ = XY + X'Z$ (YZ is redundant)**
  - Proof: $XY + X'Z + YZ = XY + X'Z + (X + X')YZ$

    $= (XY + XYZ) + (X'Z + X'YZ)$

    $= XY(1 + Z) + X'Z(1 + Y) = XY + X'Z$
- **How to find consensus terms?**
  1. Find a pair of terms, one of which contains a variable and the other contains its complement

     $A'C'D + A'\underline{BD} + BCD + A\underline{BC} + ACD'$ $(A \leftrightarrow A')$
  2. Ignore the variable and its complement; the left variables composite the consensus term

     $(A'\underline{BD}) + (A\underline{BC}) \Rightarrow \underline{BD} \bullet \underline{BC} = BCD$ (redundant term)
- **e.g.,**

  $a'b' + ac + bc' + b'c + ab = a'b' + ac + bc'$

Ordering Does Matter!

- **e.g.,** ***A'C'D + A'BD + BCD + ABC + ACD'***
  - *A'C'D + A'BD + BCD + ABC + ACD'*

    = *A'C'D + A'BD + ABC + ACD'*

    ⇒ 4 terms
  - *A'C'D + A'BD + BCD + ABC + ACD'*

    = *A'C'D + BCD + ACD'*

    ⇒ Only 3 terms!

Dual Form of the Consensus Theorem

- $XY + X'Z + YZ = XY + X'Z \Rightarrow YZ$ **is redundant**

  $\Updownarrow$ dual

- $(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z) \Rightarrow (Y + Z)$ **is redundant**

- **e.g.,**

  $$(a + b + c')(a + b + d')(b + c + d') = (a + b + c')(b + c + d')$$

  $$(a + b + c') + (b + c + d') \Rightarrow a + b + b + d' = a + b + d'$$

Redundancy Injection

- **e.g., $ABCD + B'CDE + A'B' + BCE'$**
  - Find consensus terms

    $ABCD + B'CDE \Rightarrow ACDE$

    $ABCD + A'B' \Rightarrow$ ~~$BCD(B')$~~ $+$ ~~$(ACD)A'$~~

    $B'CDE + BCE' \Rightarrow$ ~~$CDE(CE')$~~ $+$ ~~$B'CD(BC)$~~

    $A'B' + BCE' \Rightarrow A'CE'$
  - Add consensus term $ACDE$

    $ABCD + B'CDE + \underline{A}'B' + BC\underline{E}' + \underline{A}CD\underline{E}$

    $= A'B' + BCE' + ACDE$

Summary of Algebraic Simplification

Simplification can reduce cost

Rule A -- Combining Terms

- $XY + XY' = X(Y + Y') = X$
- **e.g.,**
  - $abc'd' + abcd' = abd'$ ($X = abd'$, $Y = c$)
  - $ab'c + abc + a'bc = ab'c + abc + abc + a'bc$ (repeat term)
    $= ac + bc$
  - $(a + bc)(d + e') + a'(b' + c')(d + e') = d + e'$ (why?? DeMorgan)

Rule B -- Eliminating Terms

1. **$X + XY = X$ (keep the boss)**
2. **$XY + X'Z + YZ = XY + X'Z$ (consensus)**

- **e.g.,**
  - $a'b + a'bc = a'b$ ($X = a'b$)
  - $a'bc' + bcd + a'bd = a'bc' + bcd$ ($X = c$, $Y = bd$, $Z = a'b$)

Rule C -- Eliminating Literals

- $X + X'Y = (X + X')(X + Y) = X + Y$
- **e.g.,**
  - $A'B + A'B'C'D' + ABCD'$

  $= A'(B + B'C'D') + ABCD'$ (common term: A')

  $= A'(B + C'D') + ABCD'$ (Rule C)

  $= A'B + A'C'D' + ABCD'$

  $= B(A' + ACD') + A'C'D'$ (common term: B)

  $= B(A' + CD') + A'C'D'$ (Rule C)

  $= A'B + BCD' + A'C'D'$ (final terms)

Rule D -- Adding Redundant Terms

1. $Y = Y + XX'$
2. $Y = Y(X + X')$
3. $XY + X'Z = XY + X'Z + YZ$
4. $X = X + XY$

Add redundancy to eliminate other terms

- **e.g.,**
  - $WX + XY + X'Z' + WY'Z'$

  $= WX + XY + X'Z' + WY'Z' + WZ'$ (add WZ' by consensus thm)

  $= WX + XY + X'Z' + WZ'$ (eliminate $WY'Z'$ by WZ')

  $= WX + XY + X'Z'$ (consensus again)

Another Example

- ***A'B'C'D' + A'BC'D' + A'BD + A'BC'D + ABCD + ACD' + B'CD'***

  = … (Apply rules A, B, C, D)

  = *A'C'D' + A'BD + B'CD' + ABC*

- **No easy way to determine when a Boolean expression has a minimum # of terms or literals**
  - Systematic way will be discussed in Unit 5 & Unit 6

Proving Validity of an Equation

How to Determine if an Equation Valid?

1. **Construct a truth table (proof by cases)**
2. **Manipulate one side until it is identical to the other side**
3. **Reduce both sides independently to the same expression**
4. **Perform the same operation on both sides if the operation is reversible (Boolean algebra ≠ ordinary algebra)**
   - Complement is reversible
   - Multiplication/division and addition/subtraction are not reversible
     - $x+y = x+z$ does not imply $y=z$ (e.g., $x=1$, $y=0$, $z=1$)
     - $xy = xz$ does not imply $y=z$ (e.g., $x=0$, $y=0$, $z=1$)

- **Using 2. and 3., usually…**
  1. Reduce both sides to (minimum) SOP or POS
  2. Compare both sides
  3. Try to add or delete terms by using theorems

Example

- **Show that** $A'BD' + BCD + ABC' + AB'D = BC'D' + AD + A'BC$
  - By the consensus theorem,

$$A'BD' + BCD + ABC' + AB'D$$

$$= A'BD' + BCD + ABC' + AB'D + A'BC + BC'D' + ABD$$

$$= AD + A'BD' + BCD + ABC' + A'BC + BC'D'$$

$$= AD + A'BC + BC'D'$$

One More Example

- **Show that** $A'BC'D + (A' + BC)(A + C'D') + BC'D + A'BC'$
  $= ABCD + A'C'D' + ABD + ABCD' + BC'D$

1. Reduce the left side:

   $A'BC'D + (A' + BC)(A + C'D') + BC'D + A'BC'$

   $= (A' + BC)(A + C'D') + BC'D + A'BC'$

   $= A'C'D' + ABC + BC'D + A'BC'$ (multiplying out)

   $= A'C'D' + ABC + BC'D$ (consensus)

2. Reduce the right side:

   $ABCD + A'C'D' + ABD + ABCD' + BC'D$

   $= ABC + A'C'D' + ABD + BC'D$

   $= ABC + A'C'D' + BC'D$ (consensus)

3. Because both sides were independently reduced to the same expression, the original equation is valid.